L-series from Feynman diagrams with up to 22 loops

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Beyond 3 loops, polylogarithms no longer suffice for the QED contributions to the magnetic moment of the electron. At 4 loops, one encounters an L-series of a **modular** form of weight 4. I shall report on **L-series** that similarly result from Feynman diagrams with up to **22 loops**. A salient feature is the existence of intricate **quadratic** relations between Feynman integrals, encoded by **Betti** and **de Rham** matrices that generalize Legendre's quadratic relation between elliptic integrals. Here is the plan:

1. Stefano Laporta's great feat in **4-loop** QED
2. Automorphic forms up to **6 loops**
3. L-series up to **22 loops** [*]
4. Betti and de Rham matrices for **all loops** [*]

[*] with **David P. Roberts**, University of Minnesota Morris, USA

1 Stefano Laporta's great feat in 4-loop QED

The **magnetic moment** of the electron, in Bohr magnetons, has QED contributions $\sum_{L\geq 0} a_L(\alpha/\pi)^L$ that are given up to $L = 4$ loops by

$$\begin{aligned}
a_0 &= 1 \quad [\textbf{Dirac}, 1928]\\
a_1 &= 0.5 \quad [\textbf{Schwinger}, 1947]\\
a_2 &= -0.32847896557919378458217281696489239241111929867962\ldots\\
a_3 &= 1.18124145658720000627475398221287785336878939093213\ldots\\
a_4 &= -1.91224576492644557415264716743983005406087339065872\ldots
\end{aligned}$$

In 1957, corrections by **Petermann** and **Sommerfield** resulted in

$$a_2 = \frac{197}{144} + \frac{\zeta(2)}{2} + \frac{3\zeta(3) - 2\pi^2 \log 2}{4}\,.$$

In 1996, **Laporta** and **Remiddi** [hep-ph/9602417] gave us

$$\begin{aligned}
a_3 \quad = \quad & \frac{28259}{5184} + \frac{17101\zeta(2)}{135} + \frac{139\zeta(3) - 596\pi^2 \log 2}{18}\\
& - \frac{39\zeta(4) + 400U_{3,1}}{24} - \frac{215\zeta(5) - 166\zeta(3)\zeta(2)}{24}\,.
\end{aligned}$$

The 3-loop contribution contains a weight-4 depth-2 **polylogarithm**

$$U_{3,1} := \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \frac{\zeta(4)}{2} + \frac{(\pi^2 - \log^2 2)\log^2 2}{12} - 2\sum_{n>0}\frac{1}{2^n n^4}$$

encountered in my study of **alternating** sums [arXiv:hep-th/9611004].

Equally fascinating is the **Bessel** moment $B := \sqrt{3}E_{4a}$, at weight 4, in the breath-taking evaluation by **Laporta** [arXiv:1704.06996] of **4800 digits** of

$$a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912$$

where P comprises polylogs and E comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals.
U comes from 6 light-by-light master integrals, still under investigation.

The weight-4 **non-polylogarithm** at 4 loops has $N = 6$ Bessel functions:

$$\begin{aligned} B &= -\int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t)K_0^5(t)\mathrm{d}t \\ &= -1483.68505914882529459059985184510836700500152630607810\ldots \end{aligned}$$

with 5 instances of $K_0(t)$, from 5-fermion intermediate states. The sibling of $K_0(t)$ is $I_0(t) = \sum_{k\geq 0}((t/2)^k/k!)^2$, resulting from Fourier transformation. The powers of t in B are easy to interpret in $D = 2$ spacetime dimensions.

2 Automorphic forms up to 6 loops

With $N = a + b$ **Bessel** functions and $c \geq 0$, I define **moments**

$$M(a,b,c) := \int_0^\infty I_0^a(t) K_0^b(t) t^c \mathrm{d}t$$

that converge for $b > a \geq 0$. For $b = a = N/2$, we have convergence for $b > c+1$. The L-loop on-shell **sunrise** diagram in $D = 2$ dimensions gives

$$2^L M(1, L+1, 1) = \int_0^\infty \ldots \int_0^\infty \frac{\prod_{k=1}^L \mathrm{d}x_k/x_k}{(1+\sum_{i=1}^L x_i)(1+\sum_{j=1}^L 1/x_j) - 1}$$

as an integral over Schwinger parameters. $M(2, L, 1)$ is obtained by cutting an internal line. To obtain $M(1, L+1, 3)$ and $M(2, L, 3)$, we differentiate w.r.t. an external momentum, before taking the **on-shell** limit.

2.1 3-loop sunrise at $N = 5$

In 2007, **reciprocal** PSLQ, inspired by Legendre, gave me the **matrix**

$$\mathcal{M}_5 := \left[\begin{array}{cc} M(1,4,1) & M(1,4,3) \\ M(2,3,1) & M(2,3,3) \end{array}\right] = \left[\begin{array}{cc} \pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\ \frac{\sqrt{15}\pi}{2} C & \frac{\sqrt{15}\pi}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right) \end{array}\right].$$

The **determinant** $\det \mathcal{M}_5 = 2\pi^3/\sqrt{3^3 5^5}$ is **free** of the 3-loop constant

$$C := \frac{\pi}{16}\left(1-\frac{1}{\sqrt{5}}\right)\left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4$$

that comes from the **square** of an elliptic integral [arXiv:0801.0891] at the 15th singular value. The L-series for $N=5$ Bessel functions comes from a **modular** form of weight 3 and level 15 [arXiv:1604.03057]:

$$\begin{aligned}
\eta_n &:= q^{n/24}\prod_{k>0}(1-q^{nk})\\
f_{3,15} &:= (\eta_3\eta_5)^3+(\eta_1\eta_{15})^3 = \sum_{n>0} A_5(n)q^n\\
L_5(s) &:= \sum_{n>0}\frac{A_5(n)}{n^s}\\
\Lambda_5(s) &:= \left(\frac{15}{\pi^2}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L_5(s) = \Lambda_5(3-s)\\
L_5(1) &= 5C = \frac{1}{48\sqrt{5}\pi^2}\,\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right)
\end{aligned}$$

with a product of Γ values from the Chowla-Selberg theorem.

2.2 The Laporta frontier at $N = 6$

Here the **modular** form, found with **Francis Brown** in 2010, is

$$f_{4,6} := (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$2M(3,3,1) = 3L_6(2), \quad 2M(2,4,1) = 3L_6(3), \quad 2M(1,5,1) = \pi^2 L_6(2).$$

It is notable that the hypergeometric series in

$$L_6(3) = \frac{\pi^2}{15}\, {}_4F_3\left(\begin{array}{cccc|c} \frac{1}{3}, & \frac{1}{2}, & \frac{1}{2}, & \frac{2}{3} & \\ \frac{5}{6}, & 1, & \frac{7}{6} & & 1 \end{array}\right)$$

does **not** appear in Laporta's final result, though $A_3 := 20L_6(3)/3$ appeared at intermediate stages of his calculation. Thus 4-loop QED engages only the **first** row of the **determinant** [arXiv:1604.03057]

$$\det\begin{bmatrix} M(1,5,1) & M(1,5,3) \\ M(2,4,1) & M(2,4,3) \end{bmatrix} = \frac{5\zeta(4)}{32}.$$

2.3 Kloosterman moments at $N = 7$

With $N = 7$ Bessel functions, the **local** factors at the **primes** in

$$L_7(s) = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

are given, for the **good** primes p coprime to 105, by the **cubic**

$$Z_7(p, T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |\lambda_p|^2) T + p^4 T^2\right)$$

where $(\frac{p}{105}) = \pm 1$ is a **Kronecker** symbol and λ_p is a Hecke eigenvalue of a weight-3 newform with level 525. For the primes of **bad** reduction, I obtained **quadratics** from **Kloosterman** moments in **finite fields**:

$$Z_7(3, T) = 1 - 10T + 3^4 T^2,\ Z_7(5, T) = 1 - 5^4 T^2,\ Z_7(7, T) = 1 + 70T + 7^4 T^2.$$

Then **Anton Mellit** suggested a **functional equation**

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5 - s)$$

that was validated at high precision and gave us the result

$$24M(2, 5, 1) = 5\pi^2 L_7(2).$$

2.4 Subtleties at $N = 8$

With $N = 8$ Bessel functions, the L-series comes from the **modular** form

$$f_{6,6} := \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$M(4,4,1) = L_8(3), \quad 4M(3,5,1) = 9L_8(4), \quad 4M(2,6,1) = 27L_8(5).$$

Moreover, $4M(1,7,1) = 9\pi^2 L_8(4)$ determines the **6-loop sunrise** integral.

There are **two subtleties**. First, Kloosterman moments at $N = 8$ do **not** deliver the local factors directly. In the appendix, I **remove** factors that occurred at $N = 4$. Secondly, there is an infinite family of **sum rules**:

$$A(n) := \left(\frac{2}{\pi}\right)^4 \int_0^\infty \left(\pi^2 I_0^2(t) - K_0^2(t)\right) I_0(t) K_0^5(t)\,(2t)^{2n-1}\mathrm{d}t$$

delivers the **integers** of `http://oeis.org/A262961` as was very recently proven by Yajun Zhou [http://arxiv.org/abs/1706.01068].

2.5 Vacuum integrals and non-critical modular L-series

In the **modular** cases $N = 5, 6, 8$, L-series **outside** the critical strip are related to **determinants** that involve the **vacuum** integrals $M(0, N, 1)$:

$$\begin{aligned}
\det \int_0^\infty K_0^3(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{45}{8\pi^2} L_5(4) \\
\det \int_0^\infty K_0^4(t) \left[\begin{array}{cc} K_0^2(t) & t^2 K_0^2(t) \\ I_0^2(t) & t^2 I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{27}{4\pi^2} L_6(5) \\
\det \int_0^\infty K_0^6(t) \left[\begin{array}{cc} K_0^2(t) & t^2(1-2t^2) K_0^2(t) \\ I_0^2(t) & t^2(1-2t^2) I_0^2(t) \end{array}\right] t\,\mathrm{d}t &= \frac{6075}{128\pi^2} L_8(7)\,.
\end{aligned}$$

2.6 Signpost

In work at $N > 8$ with **David Roberts** these features are notable:
local factors from **Kloosterman** moments, sometimes with adjustment;
guesses of Γ factors, signs and conductors in **functional equations**;
empirical fits of L-series to **determinants** of Feynman integrals;
quadratic relations between Bessel moments; **sum rules** when $4|N$.
We did **not** encounter modular forms or relations with vacuum integrals.

3 L-series up to 22 loops

Let $\Omega_{a,b}$ be the **determinant** of the $r \times r$ matrix with $M(a,b,1)$ at top left, size $r = \lceil (a+b)/4 - 1 \rceil$, powers of t^2 increasing to the right and powers of $I_0^2(t)$ increasing downwards. Thus $\Omega_{1,23}$ is a 5×5 determinant with the **22-loop sunrise** integral $M(1,23,1)$ at **top left** and $M(9,15,9)$ at bottom right. With $N = 4r + 4$ Bessel functions, we discovered that

$$\begin{aligned}
L_8(4) &= \frac{2^2\,\Omega_{1,7}}{3^2\pi^2} \equiv \frac{4}{9\pi^2}\int_0^\infty I_0(t)K_0^7(t)t\mathrm{d}t \\
L_{12}(6) &= \frac{2^6\,\Omega_{1,11}}{3^4 \times 5\pi^6} \\
L_{16}(8) &= \frac{2^{14}\,\Omega_{1,15}}{3^7 \times 5^2 \times 7\pi^{12}} \\
L_{20}(10) &= \frac{2^{22} \times 11 \times \mathbf{131}\,\Omega_{1,19}}{3^{11} \times 5^6 \times 7^3\pi^{20}} \qquad \text{to 44 digits} \\
L_{24}(12) &= \frac{2^{29} \times \mathbf{12558877}\,\Omega_{1,23}}{3^{19} \times 5^9 \times 7^3 \times 11\pi^{30}} \qquad \text{to 19 digits,}
\end{aligned}$$

where boldface highlights **primes** greater than N. We used **Kloosterman sums** over finite fields $\mathbf{F}_q$ with $q < 250000$. **25 GHz-years** of work, on 50 cores, gave 44-digit **precision** for $L_{20}(10)$. $L_{24}(12)$ agrees up to 19 digits.

With a **cut** of a line in the diagram at top left of the matrix, we found

$$\begin{aligned}
L_8(5) &= \frac{2^2\,\Omega_{2,6}}{3^3} \equiv \frac{4}{27}\int_0^\infty I_0^2(t)K_0^6(t)t\mathrm{d}t\\
L_{12}(7) &= \frac{2^5\times 11\,\Omega_{2,10}}{3^6\times 5^2\pi^2}\\
L_{16}(9) &= \frac{2^{14}\times 13\,\Omega_{2,14}}{3^9\times 5^3\times 7^2\pi^6}\\
L_{20}(11) &= \frac{2^{19}\times 17\times 19\times \mathbf{23}\,\Omega_{2,18}}{3^{13}\times 5^7\times 7^3\pi^{12}}\\
L_{24}(13) &= \frac{2^{27}\times 17\times 19^2\times 23^2\times \mathbf{46681}\,\Omega_{2,22}}{3^{23}\times 5^{12}\times 7^4\times 11^2\pi^{20}}\,.
\end{aligned}$$

At $N=12,16,20$, with an **odd** sign in the functional equation, we found

$$\begin{aligned}
-L_{12}'(5) &= \frac{2^4\left(2^6\times \mathbf{29}\,\widehat{\Omega}_{2,10}+3\,\Omega_{2,10}\log 2\right)}{3^2\times 7\pi^6}\\
-L_{16}'(7) &= \frac{2^9\left(2^7\times \mathbf{83}\,\widehat{\Omega}_{2,14}+3\times 11\,\Omega_{2,14}\log 2\right)}{3^5\times 5\pi^{12}}\\
-L_{20}'(9) &= \frac{2^{17}\times 17\times 19\left(2^9\times 7\times \mathbf{101}\,\widehat{\Omega}_{2,18}+5\times 13\,\Omega_{2,18}\log 2\right)}{3^8\times 5^4\times 7^2\times 11\pi^{20}}
\end{aligned}$$

for **central derivatives**, using **enlarged** determinants $\widehat{\Omega}_{2,4r+2}$ of size $r+1$ with **regularization** of $M(2r+2,2r+2,2r+1)$ at bottom right.

In the cases with $N = 4r + 2$, we obtained

$$\begin{aligned}
L_6(2) &= \frac{2\,\Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2}\int_0^\infty I_0(t)K_0^5(t)t\mathrm{d}t \quad \text{[present in a}_4\text{]}\\
L_6(3) &= \frac{2\,\Omega_{2,4}}{3} \equiv \frac{2}{3}\int_0^\infty I_0^2(t)K_0^4(t)t\mathrm{d}t \quad \text{[absent from a}_4\text{]}\\
L_{10}(4) &= \frac{2^7\,\Omega_{1,9}}{3^2\pi^6}\\
L_{10}(5) &= \frac{2^4\,\Omega_{2,8}}{3\times 5\pi^2}\\
L_{14}(6) &= 0\\
L_{14}(7) &= \frac{2^{10}\times 11\times 13\,\Omega_{2,12}}{3^6\times 5^2\times 7\pi^6}\\
L_{18}(8) &= \frac{2^{21}\times 17\times \mathbf{19}\,\Omega_{1,17}}{3^5\times 5^4\times 7\pi^{20}}\\
L_{18}(9) &= \frac{2^{12}\times 13\times 17\times \mathbf{41}\,\Omega_{2,16}}{3^8\times 5^3\times 7^2\pi^{12}}\\
L_{22}(10) &= 0\\
L_{22}(11) &= \frac{2^{23}\times 17\times 19\times \mathbf{11621}\,\Omega_{2,20}}{3^{14}\times 5^7\times 7^3\pi^{20}}
\end{aligned}$$

with central vanishing from an odd sign at $N = 14$ and $N = 22$.

For cases with odd N, we obtained

$$\begin{aligned}
L_5(2) &= \frac{2^2\,\Omega_{2,3}}{3} \equiv \frac{4}{3}\int_0^\infty I_0^2(t)K_0^3(t)t\mathrm{d}t \\
L_7(2) &= \frac{2^3\times 3\,\Omega_{2,5}}{5\pi^2} \equiv \frac{24}{5\pi^2}\int_0^\infty I_0^2(t)K_0^5(t)t\mathrm{d}t \\
L_9(4) &= \frac{2^6\,\Omega_{2,7}}{3\times 5\pi^2} \\
L_{11}(4) &= \frac{2^8\times 5\,\Omega_{2,9}}{3\times 7\pi^6} \\
L_{13}(6) &= \frac{2^7\times \mathbf{149}\,\Omega_{2,11}}{3^3\times 5\times 7\pi^6} \\
L_{15}(6) &= \frac{2^8\times 7\times \mathbf{53}\,\Omega_{2,13}}{3^2\times 5\pi^{12}} && \text{to 43 digits} \\
L_{17}(8) &= \frac{2^{15}\times \mathbf{29}\,\Omega_{2,15}}{3^5\times 5^2\times 7\pi^{12}} && \text{to 23 digits} \\
L_{19}(8) &= \frac{2^{14}\times \mathbf{1093}\times \mathbf{13171}\,\Omega_{2,17}}{3^4\times 5^4\times 7\times 11\pi^{20}} && \text{to 14 digits.}
\end{aligned}$$

Comment: We have **parallel** results relating Bessel moments $M(a,b,c)$ with **even** c to L-series with local factors obtained from Kloosterman moments with a quadratic **twist**. Quantum theory seems not to use these.

4 Betti and de Rham matrices for all loops

Construction: Let v_k and w_k be the rational numbers **generated** by

$$\begin{aligned}
\frac{J_0^2(t)}{C(t)} &= \sum_{k\geq 0} \frac{v_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{17t^2}{54} + \frac{3781t^4}{186624} + \ldots \\
\frac{2J_0(t)J_1(t)}{tC(t)} &= \sum_{k\geq 0} \frac{w_k}{k!}\left(\frac{t}{2}\right)^{2k} = 1 - \frac{41t^2}{216} + \frac{325t^4}{186624} + \ldots
\end{aligned}$$

where $J_0(t) = I_0(\mathrm{i}t)$, $J_1(t) = -J_0'(t)$ and

$$C(t) := \frac{32(1 - J_0^2(t) - tJ_0(t)J_1(t))}{3t^4} = 1 - \frac{5t^2}{27} + \frac{35t^4}{2304} - \frac{7t^6}{9600} + \ldots$$

We obtain bivariate polynomials by the **recursion**

$$\begin{aligned}
H_s(y,z) &= zH_{s-1}(y,z) - (s-1)yH_{s-2}(y,z) \\
&\quad - \sum_{k=1}^{s-1} \binom{s-1}{k} \left(v_k H_{s-k}(y,z) - w_k z H_{s-k-1}(y,z)\right)
\end{aligned}$$

for $s > 0$, with $H_0(y,z) = 1$. We use these to define

$$d_s(N,c) := \frac{H_s(3c/2, N+2-2c)}{4^s s!}.$$

Matrices: We construct rational **de Rham** matrices, with elements

$$D_N(a,b) := \sum_{c=-b}^{a} d_{a-c}(N,-c)d_{b+c}(N,c)c^{N+1}$$

and a and b running from 1 to $k = \lceil N/2-1 \rceil$.
We act on those, on the left, with **period** matrices whose elements are

$$\begin{aligned} P_{2k+1}(u,a) &:= \frac{(-1)^{a-1}}{\pi^u}M(k+1-u,k+u,2a-1) \\ P_{2k+2}(u,a) &:= \frac{(-1)^{a-1}}{\pi^{u+1/2}}M(k+1-u,k+1+u,2a-1) \end{aligned}$$

and on the right with their `transposes`, to define **Betti** matrices

$$B_N := P_N D_N P_N^{\mathtt{tr}}.$$

Conjecture 1: The Betti matrices have **rational** elements given by

$$\begin{aligned} B_{2k+1}(u,v) &= (-1)^{u+k}2^{-2k-2}(k+u)!(k+v)!Z(u+v) \\ B_{2k+2}(u,v) &= (-1)^{u+k}2^{-2k-3}(k+u+1)!(k+v+1)!Z(u+v+1) \\ Z(m) &= \frac{1+(-1)^m}{(2\pi)^m}\zeta(m). \end{aligned}$$

Comment: This gives **quadratic** relations between Feynman periods. A **parallel** conjecture for **even** Bessel moments relates to **twisted** L-series.

Conjecture 2: For integers $b > 0$ and $c \geq 0$, with $b + c$ odd,

$$S(b,c) := \left(\frac{2}{\pi}\right)^{b+1} \int_0^\infty \Im\left(\left[\pi I_0(t)K_0(t) + \mathrm{i}K_0^2(t)\right]^b\right) t^c \mathrm{d}t$$

is a **rational** number that vanishes if and only if $b > c + 1$. For $j \geq 0$,

$$Q_j(x) := 4^j j! S(2x, 2x + 2j - 1)$$

is a **monic** rational polynomial in x, of degree j, with $Q_j(0) = \delta_{j,0}$.

Comment: This gives linear relations for **odd** moments with $N = 4r + 4$. It gives relations for **even** moments with $N = 4r + 2$ in the twisted case. Both sets come from kernels of **singular** Betti and de Rahm matrices. The conjecture on $Q_j(x)$ leads to a **recursion** for determining $S(b,c)$.

Mathematical health warning

This work is highly **empirical**. Little of what I have presented for $N > 5$ is proven. It took almost a **decade** to prove some of the results at $N = 5$. **Stop press** [6 June]: Zhou's work makes proof of Conjecture 2 feasible.

Summary

1. **QED** at 4 loops involves **Bessel** moments and a weight-4 **L-series**.
2. The L-series for 5, 6 and 8 Bessel functions are **modular**. This seems to be necessary for relating **vacuum** integrals to **non**-critical L-series.
3. Relations between **determinants** of **Feynman** integrals and L-series have been discovered up to **22 loops** and presumably go on for **ever**.
4. We have **parallel** results for even moments and **twisted** L-series.
5. Legendre's relation $EK' + K(E' - K') = \pi/2$ for **elliptic** integrals foreshadows **quadratic** relations of the form $P_N D_N P_N^{\rm tr} = B_N$ with **period**, **de Rham** and **Betti** matrices that we have specified.
6. With $N = 4r + 4$ Bessel functions, the **kernels** of the singular matrices B_N and D_N give **linear** relations between Feynman periods.

I thank skilful colleagues and generous hosts at recent meetings in Creswick (Victoria), Newcastle (NSW), Mainz and Oxford for kind help.

Appendix: Kloosterman sums over finite fields

For $a \in \mathbf{F}_q$, with $q = p^k$, we define Kloosterman sums

$$K(a) := \sum_{x \in \mathbf{F}_q^*} \exp\left(\frac{2\pi \mathrm{i}}{p} \mathrm{Trace}\left(x + \frac{a}{x}\right)\right)$$

with a trace of Frobenius in $\mathbf{F}_q$ over $\mathbf{F}_p$. Then we obtain integers

$$c_N(q) := -\frac{1 + S_N(q)}{q^2}$$

$$S_N(q) := \sum_{a \in \mathbf{F}_q^*} \sum_{k=0}^{N} [g(a)]^k [h(a)]^{N-k}$$

with $K(a) = -g(a) - h(a)$ and $g(a)h(a) = q$. Then

$$Z_N(p, T) = \exp\left(-\sum_{k>0} \frac{c_N(p^k)}{k} T^k\right)$$

is a polynomial in T. For $N < 8$, the appropriate L-series is

$$L_N(s) = \prod_p \frac{1}{Z_N(p, p^{-s})}$$

with a modification at $N = 8$:

$$L_8(s) = \prod_p \frac{Z_4(p, p^{2-s})}{Z_8(p, p^{-s})}\,.$$