Beyond 3 loops, polylogarithms no longer suffice for the QED contributions to the magnetic moment of the electron. At 4 loops, one encounters an L-series of a modular form of weight 4. I shall report on L-series that similarly result from Feynman diagrams with up to 22 loops. A salient feature is the existence of intricate quadratic relations between Feynman integrals, encoded by Betti and de Rham matrices that generalize Legendre’s quadratic relation between elliptic integrals. Here is the plan:

1. Stefano Laporta’s great feat in 4-loop QED
2. Automorphic forms up to 6 loops
3. L-series up to 22 loops [*]
4. Betti and de Rham matrices for all loops [*]

[*] with David P. Roberts, University of Minnesota Morris, USA
1 Stefano Laporta’s great feat in 4-loop QED

The magnetic moment of the electron, in Bohr magnetons, has QED contributions \( \sum_{L \geq 0} a_L (\alpha/\pi)^L \) that are given up to \( L = 4 \) loops by

\[

tag{1} a_0 = 1 \quad [\text{Dirac, 1928}]
tag{2} a_1 = 0.5 \quad [\text{Schwinger, 1947}]
tag{3} a_2 = -0.32847896557919378458217281696489239241111929867962\ldots
\]
\[
tag{4} a_3 = 1.18124145658720000627475398221287785336878939093213\ldots
\]
\[
tag{5} a_4 = -1.91224576492644557415264716743983005406087339065872\ldots
\]

In 1957, corrections by Petermann and Sommerfield resulted in

\[

tag{6} a_2 = \frac{197}{144} + \frac{\zeta(2)}{2} + \frac{3\zeta(3) - 2\pi^2 \log 2}{4}.
\]


\[

tag{7} a_3 = \frac{28259}{5184} + \frac{17101\zeta(2)}{135} + \frac{139\zeta(3) - 596\pi^2 \log 2}{18} - \frac{39\zeta(4) + 400U_{3,1}}{24} - \frac{215\zeta(5) - 166\zeta(3)\zeta(2)}{24}.
\]
The 3-loop contribution contains a weight-4 depth-2 polylogarithm

\[ U_{3,1} := \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3n} = \frac{\zeta(4)}{2} + \frac{(\pi^2 - \log^2 2) \log^2 2}{12} - 2 \sum_{n>0} \frac{1}{2^n n^4} \]

encountered in my study of alternating sums [arXiv:hep-th/9611004].

Equally fascinating is the Bessel moment \( B := \sqrt{3} E_{4a} \), at weight 4, in the breath-taking evaluation by Laporta [arXiv:1704.06996] of 4800 digits of

\[ a_4 = P + B + E + U \approx 2650.565 - 1483.685 - 1036.765 - 132.027 \approx -1.912 \]

where \( P \) comprises polylogs and \( E \) comprises integrals, with weights 5, 6 and 7, whose integrands contain logs and products of elliptic integrals. \( U \) comes from 6 light-by-light master integrals, still under investigation.

The weight-4 non-polylogarithm at 4 loops has \( N = 6 \) Bessel functions:

\[
B = - \int_0^\infty \frac{27550138t + 35725423t^3}{48600} I_0(t) K_0^5(t) dt = -1483.68505914882529459059985184510836700500152630607810 \ldots
\]

with 5 instances of \( K_0(t) \), from 5-fermion intermediate states. The sibling of \( K_0(t) \) is \( I_0(t) = \sum_{k \geq 0} ((t/2)^k / k!)^2 \), resulting from Fourier transformation. The powers of \( t \) in \( B \) are easy to interpret in \( D = 2 \) spacetime dimensions.
2 Automorphic forms up to 6 loops

With \( N = a + b \) Bessel functions and \( c \geq 0 \), I define \textbf{moments}

\[
M(a, b, c) := \int_0^\infty I^a_0(t)K^b_0(t)t^c \, dt
\]

that converge for \( b > a \geq 0 \). For \( b = a = N/2 \), we have convergence for \( b > c + 1 \). The \( L \)-loop on-shell \textbf{sunrise} diagram in \( D = 2 \) dimensions gives

\[
2^LM(1, L + 1, 1) = \int_0^\infty \cdots \int_0^\infty \frac{\prod_{k=1}^L dx_k/x_k}{(1 + \sum_{i=1}^L x_i)(1 + \sum_{j=1}^L 1/x_j) - 1}
\]

as an integral over Schwinger parameters. \( M(2, L, 1) \) is obtained by cutting an internal line. To obtain \( M(1, L + 1, 3) \) and \( M(2, L, 3) \), we differentiate w.r.t. an external momentum, before taking the \textbf{on-shell} limit.

2.1 3-loop sunrise at \( N = 5 \)

In 2007, \textbf{reciprocal} PSLQ, inspired by Legendre, gave me the \textbf{matrix}

\[
\mathcal{M}_5 := \begin{bmatrix}
M(1, 4, 1) & M(1, 4, 3) \\
M(2, 3, 1) & M(2, 3, 3)
\end{bmatrix} = \begin{bmatrix}
\pi^2 C & \pi^2 \left(\frac{2}{15}\right)^2 \left(13C - \frac{1}{10C}\right) \\
\frac{\sqrt{15\pi}}{2} C & \frac{\sqrt{15\pi}}{2} \left(\frac{2}{15}\right)^2 \left(13C + \frac{1}{10C}\right)
\end{bmatrix}.
\]
The determinant $\det \mathcal{M}_5 = 2\pi^3/\sqrt{3^{35}}$ is free of the 3-loop constant

$$C := \frac{\pi}{16} \left(1 - \frac{1}{\sqrt{5}}\right) \left(\sum_{n=-\infty}^{\infty} e^{-n^2\pi\sqrt{15}}\right)^4$$

that comes from the square of an elliptic integral [arXiv:0801.0891] at the 15th singular value. The L-series for $N = 5$ Bessel functions comes from a modular form of weight 3 and level 15 [arXiv:1604.03057]:

$$\eta_n := q^{n/24} \prod_{k>0} (1 - q^{nk})$$

$$f_{3,15} := (\eta_3 \eta_5)^3 + (\eta_1 \eta_{15})^3 = \sum_{n>0} A_5(n) q^n$$

$$L_5(s) := \sum_{n>0} \frac{A_5(n)}{n^s}$$

$$\Lambda_5(s) := \left(\frac{15}{\pi^2}\right)^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s + 1}{2}\right) L_5(s) = \Lambda_5(3-s)$$

$$L_5(1) = 5C = \frac{1}{48\sqrt{5\pi^2}} \Gamma\left(\frac{1}{15}\right) \Gamma\left(\frac{2}{15}\right) \Gamma\left(\frac{4}{15}\right) \Gamma\left(\frac{8}{15}\right)$$

with a product of $\Gamma$ values from the Chowla-Selberg theorem.
2.2 The Laporta frontier at $N = 6$

Here the modular form, found with Francis Brown in 2010, is

$$f_{4,6} := (\eta_1 \eta_2 \eta_3 \eta_6)^2$$

with weight 4 and level 6. I discovered and checked to 1000 digits that

$$2M(3, 3, 1) = 3L_6(2), \quad 2M(2, 4, 1) = 3L_6(3), \quad 2M(1, 5, 1) = \pi^2 L_6(2).$$

It is notable that the hypergeometric series in

$$L_6(3) = \frac{\pi^2}{15} _4 \! F_3 \left( \begin{array}{c} \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3} \\ \frac{5}{6}, 1, \frac{7}{6} \end{array} \bigg| 1 \right)$$

does not appear in Laporta’s final result, though $A_3 := 20L_6(3)/3$ appeared at intermediate stages of his calculation. Thus 4-loop QED engages only the first row of the determinant [arXiv:1604.03057]

$$\det \begin{bmatrix} M(1, 5, 1) & M(1, 5, 3) \\ M(2, 4, 1) & M(2, 4, 3) \end{bmatrix} = \frac{5\zeta(4)}{32}.$$
2.3 Kloosterman moments at $N = 7$

With $N = 7$ Bessel functions, the local factors at the primes in

$$L_7(s) = \prod_p \frac{1}{Z_7(p, p^{-s})}$$

are given, for the good primes $p$ coprime to 105, by the cubic

$$Z_7(p, T) = \left(1 - \left(\frac{p}{105}\right) p^2 T\right) \left(1 + \left(\frac{p}{105}\right) (2p^2 - |\lambda_p|^2)T + p^4 T^2\right)$$

where $\left(\frac{p}{105}\right) = \pm 1$ is a Kronecker symbol and $\lambda_p$ is a Hecke eigenvalue of a weight-3 newform with level 525. For the primes of bad reduction, I obtained quadratics from Kloosterman moments in finite fields:

$$Z_7(3, T) = 1 - 10T + 3^4 T^2, \quad Z_7(5, T) = 1 - 5^4 T^2, \quad Z_7(7, T) = 1 + 70T + 7^4 T^2.$$  

Then Anton Mellit suggested a functional equation

$$\Lambda_7(s) := \left(\frac{105}{\pi^3}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L_7(s) = \Lambda_7(5-s)$$

that was validated at high precision and gave us the result

$$24M(2, 5, 1) = 5\pi^2 L_7(2).$$
2.4 Subtleties at $N = 8$

With $N = 8$ Bessel functions, the L-series comes from the **modular** form

$$f_{6,6} := \left( \frac{\eta_2^3 \eta_3^3}{\eta_1 \eta_6} \right)^3 + \left( \frac{\eta_1^3 \eta_6^3}{\eta_2 \eta_3} \right)^3$$

with weight 6 and level 6. I discovered and checked to 1000 digits that

$$M(4, 4, 1) = L_8(3), \quad 4M(3, 5, 1) = 9L_8(4), \quad 4M(2, 6, 1) = 27L_8(5).$$

Moreover, $4M(1, 7, 1) = 9\pi^2 L_8(4)$ determines the **6-loop sunrise** integral. There are two subtleties. First, Kloosterman moments at $N = 8$ do **not** deliver the local factors directly. In the appendix, I **remove** factors that occurred at $N = 4$. Secondly, there is an infinite family of **sum rules**:

$$A(n) := \left( \frac{2}{\pi} \right)^4 \int_0^\infty \left( \pi^2 I_0^2(t) - K_0^2(t) \right) I_0(t)K_0(t) (2t)^{2n-1}dt$$

delivers the integers of [http://oeis.org/A262961](http://oeis.org/A262961) as was very recently proven by Yajun Zhou [http://arxiv.org/abs/1706.01068].
2.5 Vacuum integrals and non-critical modular L-series

In the modular cases \( N = 5, 6, 8 \), L-series outside the critical strip are related to determinants that involve the vacuum integrals \( M(0, N, 1) \):

\[
\det \int_0^\infty K^3_0(t) \begin{bmatrix}
  K^2_0(t) & t^2K^2_0(t) \\
  I^2_0(t) & t^2I^2_0(t)
\end{bmatrix} \, t \, dt = \frac{45}{8\pi^2} L_5(4)
\]

\[
\det \int_0^\infty K^4_0(t) \begin{bmatrix}
  K^2_0(t) & t^2K^2_0(t) \\
  I^2_0(t) & t^2I^2_0(t)
\end{bmatrix} \, t \, dt = \frac{27}{4\pi^2} L_6(5)
\]

\[
\det \int_0^\infty K^6_0(t) \begin{bmatrix}
  K^2_0(t) & t^2(1-2t^2)K^2_0(t) \\
  I^2_0(t) & t^2(1-2t^2)I^2_0(t)
\end{bmatrix} \, t \, dt = \frac{6075}{128\pi^2} L_8(7).
\]

2.6 Signpost

In work at \( N > 8 \) with David Roberts these features are notable:
- local factors from Kloosterman moments, sometimes with adjustment;
- guesses of \( \Gamma \) factors, signs and conductors in functional equations;
- empirical fits of L-series to determinants of Feynman integrals;
- quadratic relations between Bessel moments; sum rules when \( 4 | N \).
- We did not encounter modular forms or relations with vacuum integrals.
3 L-series up to 22 loops

Let $\Omega_{a,b}$ be the determinant of the $r \times r$ matrix with $M(a, b, 1)$ at top left, size $r = \lceil (a + b)/4 - 1 \rceil$, powers of $t^2$ increasing to the right and powers of $I_0^2(t)$ increasing downwards. Thus $\Omega_{1,23}$ is a $5 \times 5$ determinant with the 22-loop sunrise integral $M(1, 23, 1)$ at top left and $M(9, 15, 9)$ at bottom right. With $N = 4r + 4$ Bessel functions, we discovered that

$$L_8(4) = \frac{2^2 \Omega_{1,7}}{3^2 \pi^2} \equiv \frac{4}{9 \pi^2} \int_0^\infty I_0(t)K_0^7(t)tdt$$

$$L_{12}(6) = \frac{2^6 \Omega_{1,11}}{3^4 \times 5\pi^6}$$

$$L_{16}(8) = \frac{2^{14} \Omega_{1,15}}{3^7 \times 5^2 \times 7\pi^{12}}$$

$$L_{20}(10) = \frac{2^{22} \times 11 \times 131 \Omega_{1,19}}{3^{11} \times 5^6 \times 7^3 \pi^{20}} \text{ to } 44 \text{ digits}$$

$$L_{24}(12) = \frac{2^{29} \times 12558877 \Omega_{1,23}}{3^{19} \times 5^9 \times 7^3 \times 11\pi^{30}} \text{ to } 19 \text{ digits},$$

where boldface highlights primes greater than $N$. We used Kloosterman sums over finite fields $F_q$ with $q < 250000$. 25 GHz-years of work, on 50 cores, gave 44-digit precision for $L_{20}(10)$. $L_{24}(12)$ agrees up to 19 digits.
With a cut of a line in the diagram at top left of the matrix, we found

\[
L_8(5) = \frac{2^2 \Omega_{2,6}}{3^3} \equiv \frac{4}{27} \int_0^\infty I_0^2(t)K_0^6(t)tdt
\]

\[
L_{12}(7) = \frac{2^5 \times 11 \Omega_{2,10}}{3^6 \times 5^2\pi^2}
\]

\[
L_{16}(9) = \frac{2^{14} \times 13 \Omega_{2,14}}{3^9 \times 5^3 \times 7^2\pi^6}
\]

\[
L_{20}(11) = \frac{2^{19} \times 17 \times 19 \times 23 \Omega_{2,18}}{3^{13} \times 5^7 \times 7^3\pi^{12}}
\]

\[
L_{24}(13) = \frac{2^{27} \times 17 \times 19^2 \times 23^2 \times 46681 \Omega_{2,22}}{3^{23} \times 5^{12} \times 7^4 \times 11^2\pi^{20}}.
\]

At \( N = 12, 16, 20 \), with an odd sign in the functional equation, we found

\[
-L'_{12}(5) = \frac{2^4 \left(2^6 \times 29 \Omega_{2,10} + 3 \Omega_{2,10} \log 2\right)}{3^2 \times 7\pi^6}
\]

\[
-L'_{16}(7) = \frac{2^9 \left(2^7 \times 83 \Omega_{2,14} + 3 \times 11 \Omega_{2,14} \log 2\right)}{3^5 \times 5\pi^{12}}
\]

\[
-L'_{20}(9) = \frac{2^{17} \times 17 \times 19 \left(2^9 \times 7 \times 101 \Omega_{2,18} + 5 \times 13 \Omega_{2,18} \log 2\right)}{3^8 \times 5^4 \times 7^2 \times 11\pi^{20}}
\]

for central derivatives, using enlarged determinants \( \Omega_{2,4r+2} \) of size \( r + 1 \) with regularization of \( M(2r + 2, 2r + 2, 2r + 1) \) at bottom right.
In the cases with $N = 4r + 2$, we obtained

\[L_6(2) = \frac{2 \Omega_{1,5}}{\pi^2} \equiv \frac{2}{\pi^2} \int_0^\infty I_0(t)K_0^5(t)tdt \quad \text{[present in a}_4\text{]}\]

\[L_6(3) = \frac{2 \Omega_{2,4}}{3} \equiv \frac{2}{3} \int_0^\infty I_0^2(t)K_0^4(t)tdt \quad \text{[absent from a}_4\text{]}\]

\[L_{10}(4) = \frac{2^7 \Omega_{1,9}}{32\pi^6}\]

\[L_{10}(5) = \frac{2^4 \Omega_{2,8}}{3 \times 5^2}\]

\[L_{14}(6) = 0\]

\[L_{14}(7) = \frac{2^{10} \times 11 \times 13 \Omega_{2,12}}{3^6 \times 5^2 \times 7\pi^6}\]

\[L_{18}(8) = \frac{2^{21} \times 17 \times 19 \Omega_{1,17}}{3^5 \times 5^4 \times 7\pi^{20}}\]

\[L_{18}(9) = \frac{2^{12} \times 13 \times 17 \times 41 \Omega_{2,16}}{3^8 \times 5^3 \times 7^2\pi^{12}}\]

\[L_{22}(10) = 0\]

\[L_{22}(11) = \frac{2^{23} \times 17 \times 19 \times 11621 \Omega_{2,20}}{3^{14} \times 5^7 \times 7^3\pi^{20}}\]

with central vanishing from an odd sign at $N = 14$ and $N = 22$. 
For cases with odd $N$, we obtained

\[
\begin{align*}
L_5(2) & = \frac{2^2 \Omega_{2,3}}{3} = \frac{4}{3} \int_0^\infty I_0^2(t)K_0^3(t)tdt \\
L_7(2) & = \frac{2^3 \times 3 \Omega_{2,5}}{5\pi^2} = \frac{24}{5\pi^2} \int_0^\infty I_0^2(t)K_0^5(t)tdt \\
L_9(4) & = \frac{2^6 \Omega_{2,7}}{3 \times 5\pi^2} \\
L_{11}(4) & = \frac{2^8 \times 5 \Omega_{2,9}}{3 \times 7\pi^6} \\
L_{13}(6) & = \frac{2^7 \times 149 \Omega_{2,11}}{3^3 \times 5 \times 7\pi^6} \\
L_{15}(6) & = \frac{2^8 \times 7 \times 53 \Omega_{2,13}}{3^2 \times 5\pi^{12}} \quad \text{to 43 digits} \\
L_{17}(8) & = \frac{2^{15} \times 29 \Omega_{2,15}}{3^5 \times 5^2 \times 7\pi^{12}} \quad \text{to 23 digits} \\
L_{19}(8) & = \frac{2^{14} \times 1093 \times 13171 \Omega_{2,17}}{3^4 \times 5^4 \times 7 \times 11\pi^{20}} \quad \text{to 14 digits.}
\end{align*}
\]

**Comment:** We have parallel results relating Bessel moments $M(a, b, c)$ with even $c$ to L-series with local factors obtained from Kloosterman moments with a quadratic twist. Quantum theory seems not to use these.
4  Betti and de Rham matrices for all loops

Construction: Let $v_k$ and $w_k$ be the rational numbers generated by

$$\frac{J_0^2(t)}{C(t)} = \sum_{k \geq 0} \frac{v_k}{k!} \left(\frac{t}{2}\right)^{2k} = 1 - \frac{17t^2}{54} + \frac{3781t^4}{186624} + \ldots$$

$$\frac{2J_0(t)J_1(t)}{tC(t)} = \sum_{k \geq 0} \frac{w_k}{k!} \left(\frac{t}{2}\right)^{2k} = 1 - \frac{41t^2}{216} + \frac{325t^4}{186624} + \ldots$$

where $J_0(t) = I_0(it)$, $J_1(t) = -J'_0(t)$ and

$$C(t) := \frac{32(1 - J_0^2(t) - tJ_0(t)J_1(t))}{3t^4} = 1 - \frac{5t^2}{27} + \frac{35t^4}{2304} - \frac{7t^6}{9600} + \ldots$$

We obtain bivariate polynomials by the recursion

$$H_s(y, z) = zH_{s-1}(y, z) - (s - 1)yH_{s-2}(y, z)$$

$$- \sum_{k=1}^{s-1} \binom{s-1}{k} (v_kH_{s-k}(y, z) - w_kzH_{s-k-1}(y, z))$$

for $s > 0$, with $H_0(y, z) = 1$. We use these to define

$$d_s(N, c) := \frac{H_s(3c/2, N + 2 - 2c)}{4^s s!}.$$
Matrices: We construct rational de Rham matrices, with elements

\[ D_N(a, b) := \sum_{c=-b}^{a} d_{a-c}(N, -c)d_{b+c}(N, c)c^{N+1} \]

and \( a \) and \( b \) running from 1 to \( k = \lceil N/2 - 1 \rceil \).
We act on those, on the left, with period matrices whose elements are

\[ P_{2k+1}(u, a) := \frac{(-1)a^{-1}}{\pi^u} M(k + 1 - u, k + u, 2a - 1) \]
\[ P_{2k+2}(u, a) := \frac{(-1)a^{-1}}{\pi^{u+1/2}} M(k + 1 - u, k + 1 + u, 2a - 1) \]

and on the right with their transposes, to define Betti matrices

\[ B_N := P_N D_N P_N^{t} \].

Conjecture 1: The Betti matrices have rational elements given by

\[ B_{2k+1}(u, v) = (-1)^{u+k} 2^{-2k-2}(k + u)!(k + v)!Z(u + v) \]
\[ B_{2k+2}(u, v) = (-1)^{u+k} 2^{-2k-3}(k + u + 1)!(k + v + 1)!Z(u + v + 1) \]
\[ Z(m) = \frac{1 + (-1)^m}{(2\pi)^m} \zeta(m) \]

Comment: This gives quadratic relations between Feynman periods. A parallel conjecture for even Bessel moments relates to twisted L-series.
Conjecture 2: For integers \(b > 0\) and \(c \geq 0\), with \(b + c\) odd,

\[
S(b, c) := \left(\frac{2}{\pi}\right)^{b+1} \int_0^\infty \Im \left( [\pi I_0(t) K_0(t) + i K_0^2(t)]^b \right) t^c \, dt
\]

is a \textbf{rational} number that vanishes if and only if \(b > c + 1\). For \(j \geq 0\),

\[
Q_j(x) := 4^j j! S(2x, 2x + 2j - 1)
\]

is a \textbf{monic} rational polynomial in \(x\), of degree \(j\), with \(Q_j(0) = \delta_{j,0}\).

\textbf{Comment:} This gives linear relations for \textbf{odd} moments with \(N = 4r + 4\). It gives relations for \textbf{even} moments with \(N = 4r + 2\) in the twisted case. Both sets come from kernels of \textbf{singular} Betti and de Rahm matrices. The conjecture on \(Q_j(x)\) leads to a \textbf{recursion} for determining \(S(b, c)\).

\textbf{Mathematical health warning}

This work is highly \textbf{empirical}. Little of what I have presented for \(N > 5\) is proven. It took almost a \textbf{decade} to prove some of the results at \(N = 5\).

\textbf{Stop press} [6 June]: Zhou’s work makes proof of Conjecture 2 feasible.
Summary

1. **QED** at 4 loops involves **Bessel** moments and a weight-4 **L-series**.

2. The L-series for 5, 6 and 8 Bessel functions are **modular**. This seems to be necessary for relating **vacuum** integrals to **non**-critical L-series.

3. Relations between **determinants** of **Feynman** integrals and L-series have been discovered up to **22 loops** and presumably go on for **ever**.

4. We have **parallel** results for even moments and **twisted** L-series.

5. Legendre’s relation $EK' + K(E' - K') = \pi/2$ for **elliptic** integrals foreshadows **quadratic** relations of the form $P_N D_N P_N^{tr} = B_N$ with **period**, **de Rham** and **Betti** matrices that we have specified.

6. With $N = 4r + 4$ Bessel functions, the **kernels** of the singular matrices $B_N$ and $D_N$ give **linear** relations between Feynman periods.

I thank skilful colleagues and generous hosts at recent meetings in Creswick (Victoria), Newcastle (NSW), Mainz and Oxford for kind help.
Appendix: Kloosterman sums over finite fields

For \( a \in \mathbb{F}_q \), with \( q = p^k \), we define Kloosterman sums

\[
K(a) := \sum_{x \in \mathbb{F}_q^*} \exp \left( \frac{2\pi i}{p} \text{Trace} \left( x + \frac{a}{x} \right) \right)
\]

with a trace of Frobenius in \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Then we obtain integers

\[
c_N(q) := -\frac{1 + S_N(q)}{q^2}
\]

\[
S_N(q) := \sum_{a \in \mathbb{F}_q^*} \sum_{k=0}^{N} [g(a)]^k [h(a)]^{N-k}
\]

with \( K(a) = -g(a) - h(a) \) and \( g(a)h(a) = q \). Then

\[
Z_N(p, T) = \exp \left( -\sum_{k>0} \frac{c_N(p^k)}{k} T^k \right)
\]

is a polynomial in \( T \). For \( N < 8 \), the appropriate L-series is

\[
L_N(s) = \prod_p \frac{1}{Z_N(p, p^{-s})}
\]

with a modification at \( N = 8 \):

\[
L_8(s) = \prod_p \frac{Z_4(p, p^{2-s})}{Z_8(p, p^{-s})}.
\]