R-, R^{-1}- and R^*-operations and multiloop RG calculations

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Main topic of the talk:

discussion of the current status of the multiloop RG calculations and methods in QCD and general gauge theories (mainly their ***algebraic*** **aspects) in connection to the classical Bogolyubov-Parasiyuk R-operation and its generalizations (R^{-1}- and R^*-operations)**

MULTI-LOOP-2017, UPMC, Paris, 07.06.2017

GGT/QCD Running Today: FIVE-loop level has been done!
(after $\approx$ 20 Years since the first 4-loop results)

- **fermion(quark) mass anomalous dimension**

P. Baikov, K.Ch. and J. Kühn (2013 /QCD_0/ , 2017 /GGT_0/)

T. Luthe, A. Maier and P. Marquard (2016, /GGT_M/)

- **β-function**

P. Baikov, K.Ch. and J. Kühn (2016 /QCD_0/)

F. Herzog, B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt (2017, /GGT_0/)

- **Both approaches, the massless and the massive ones would be not possible without sophisticated use of the R-operation + (only for the massless approach) its relatives R^* and R^{-1}**

Consider a simple theory, the ϕ^4-model with the Lagrangian

$$\mathcal{L}^r = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$

and let

$$\Gamma[\mathcal{L},\phi] = S_{cl} \quad + \delta\Gamma[\mathcal{L},\phi] \ \ \textbf{with} \ \ S_{cl} \quad = \int \mathcal{L}(\phi(x))dx$$

is the the generating functional of all 1PI Green functions corresponding to the Lagrangian $\mathcal{L}$

Renormalizability

$\Downarrow$

$$\Gamma_R[\phi] \equiv \Gamma[\mathcal{L}^R,\phi] \equiv \Gamma[\mathcal{L}^B,\phi_B]|_{\phi_B=\sqrt{Z_2}\phi}$$

with $g_B = Z_4/Z_2^2 g,\ \ m_B = Z_m m$

$$\mathcal{L}^R = \frac{1}{2}Z_2(\partial\phi)^2 - \frac{1}{2}Z_m Z_2 m^2\phi^2 - \frac{g}{4!}Z_4\phi^4$$

$$\mathcal{L}^B = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\phi^2 - \frac{g}{4!}\phi^4$$

produces finite (after regularization is removed) Green functions in every order of PT in the coupling g.

To prove the renormalzability Bogolyubov and Parasyuk invented the R-operation. Let'remind some definitions.

Let $<\Gamma>$ be a Feynman integral (FI) corresponding to a diagram Γ, then R-operation is defined as

$$R<\Gamma>=\sum_{\gamma_1,\ldots\gamma_j}\prod_i \Delta(\gamma_i)<\Gamma>=R'<\Gamma>+\Delta(\Gamma)<\Gamma>$$

1. sum goes over all sets $\{\gamma_1,\ldots\gamma_j\}$ of (pairwise) disjoint 1PI subgraphs, with $\Delta(\emptyset)=1$

2. $\Delta(\gamma)$ is a counterterm (c-) operation which acts as follows:

$$\Delta(\gamma)<\Gamma>=P_\gamma*<\Gamma/\gamma>$$

3. $P_\gamma=\Delta<\gamma>$ is a polynomial in external momenta (mandatory) and masses (desirable) of FI $<\gamma>$ which is inserted into the vertex v_γ inside of the reduced graph Γ/γ

4. a specific choice of the c-operator $\longleftrightarrow$ choice of a renormalization scheme

Main (analytical) theorem of the R-operation:

If FI $< \Gamma >$ does not contain IR divergences then the *renormalized* FI $R < \Gamma >$ is finite in the limit of removed *UV regularization* with a proper choice of the c-operation. For instance, for the Dim. reg. + $\overline{\text{MS}}$-scheme

$$\Delta(\gamma) < \gamma >= -K_\epsilon \, R' < \gamma >, \quad K_\epsilon \Big[\sum_i f_i \, \epsilon^i \Big] = \sum_{i<0} f_i \, \epsilon^i$$

In terms of the R-operation the generating functional of the renormalized Green function is written as:

$$\Gamma_R[\phi] = \Gamma[\mathcal{L}^R, \phi] \equiv R \, \Gamma[\mathcal{L}, \phi]$$

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There should be a connection between the Lagrangian with Z-factors $\mathcal{L}^R$ and R-operation (both lead to the same renormalized Green functions). It is given by

Main (combinatorial) theorem of the R-operation:

$$\mathcal{L}^c \equiv S_{cl} + \Delta\,\delta\Gamma[\mathcal{L},\phi] \equiv \Delta\,\Gamma[\mathcal{L},\phi] \quad (\textbf{assuming} \quad \Delta\,S_{cl} \equiv S_{cl})$$

The theorem

+ commutativity of K_ϵ (and, hence, the very c-operation Δ) with differentiations w.r.t. masess and (external) momenta /Collins (1975)/

+ (resulting from it) polynomiality of ($\overline{\text{MS}}$) UV counterterms in momenta and masses

provides us with a very convenient and flexible way of computing of contributions to Z-factors from *separate diagrams*.

It is the starting point of the "IR reduction"

/A. Vladimirov (1978), K. Ch., V. Smirnov (1983), . . . /

which is a method to reduce the general task of evaluation of (L+1)-loop UV counterterms to a well-defined and clearly posed purely mathematical problem: the calculation of L-loop p-integrals (that is massless propagator-type Feynman integrals) $\equiv$ L-loop Problem

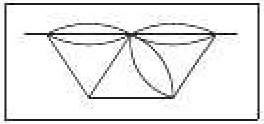
L-loop Problem: from L=2 to L=4 in about forty years

1. **1-loop Problem is trivial**

2. **the 2-loop Problem was solved after inventing and developing the Gegenbauer polynomial technique in x-space (GPTX) K.Ch.,F. Tkachov(1979)/, later /D. Broadhurst, . . . A. Kotikov/**

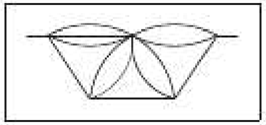
An impressive example of GPTX in action (8 loops!!)

$\bar{I}_{8d}$ =



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{2048\,\varepsilon^4}+\frac{1}{192\,\varepsilon^3}-\frac{1}{64\,\varepsilon^2}-\frac{11}{192\,\varepsilon}\right),$$

$\bar{I}_{8e}$ =



$$= \frac{1}{(4\pi)^8}\left(-\frac{1}{6144\,\varepsilon^4}+\frac{1}{256\,\varepsilon^3}-\frac{19}{384\,\varepsilon^2}+\frac{5}{16\,\varepsilon}\right).$$

For their evaluation, we used the *Gegenbauer polynomials x-space technique* (GPXT). We briefly review this technique in Appendix C.

+ many more similar integrals

copied from "The Leading Order Dressing Phase in ABJM Theory" Andrea Mauri, Alberto Santambrogio, Stefano Scoleri, arXiv:1301.7732 [hep-th]

The main breakthrough at the three loop level happened with elaborating the method of integration by parts (IBP) of **dimensionally regulated** integrals.

Historical references:

1 loop: **At one loop, IBP (for DR integrals) was used in Hooft and M. Veltman (1979)**

2 loops: crucial step **— an appropriate modification of the** $integrand$ **before IBP was first made by A. Vasiliev, Yu. Pis'mak and Yu. Khonkonen (1981)**

3 loops (complete algorithm): **F. Tkachov (1981); K. Ch. and F. Tkachov (1981)**

3-Loop Problem

With the use of IBP identities the 3-loop Problem was completely solved and corresponding (manually constructed) algorithm was effectively implemented first in SCHOONSCHIP CAS (Gorishny, Larin, Surguladze, and Tkachov) and then with FORM (Vermaseren, Larin, Tkachov, /1991/ ... Vermaseren 2000–2012).

Note that all (notrivial) masters for MINCER were provided by GPTX.

This achievement resulted to a host of various important 3- and 4-loop calculations performed by different teams during 80-th and 90-th.

Note that the 4-loop correction to the QCD β function was done only as late as in 1996 and using "massive" way /van Ritbergen, Vermaseren,and Larin/; the reason was too complicated combinatorics of the IR reduction (see later).

4-Loop Problem

It tooks around 15 years for the full solution (reduction + masters) of the 4-Loop Problem (2001 -2017). Main pieces of the solution:

- **the Baikov's way of doing reduction with the help of $1/D$ expansion of the corresponding coefficient functions in front of masters (2001-2006) /implemented as FORM-program BAICER/**

- **all 28 masters have been performed analytically with three(!) independent calculations**
  K.Ch, P. Baikov (2010); /IB+ + gluing/;
  R. Lee, V. Smirnov (2012) /IBP + reccurence relation w.r.t. D + /
  E. Panzer (2013) /no IBP, "direct" parametrtic integration with hyperlogs/

- **reduction rules automatically derived by LitRed (R. Lee (2014)) (their effective implementaion is not trivial!)**

- **any sufficiently powefull Laporta machine (e.g. FIRE by A. Smirnov (2008-. . .) or, CRUSHER by P. Marquard and D. Seidel, (2004- . . .) or . . .)**

Last but NOT the least:

Forcer, a FORM program for the parametric reduction of four-loop massless propagator diagrams,

B. Ruijl, T. Ueda, J.A.M. Vermaseren, arXiv:1704.06650

- The FORM program performs parametric IBP reductions similar to the three-loop Mincer program.
- Unlike many others tools it works “out of the box”, even topologies are recognized automatically
- FORCER is publically available (unlike BAICER and CRUSHER)
- First calculations demonstrate that with FORCER one could do 4-loop propagators *significantly faster* than, say, with BAICER and, probably, the current Laporta-based tools (with characteristic times weeks, not years . . .)

Starting point of Infrared Reduction /A. Vladimirov (1978)/

Let $<\Gamma>$ be a (L+1)-loop diagram with UV index of divergency $\omega(\Gamma) = 0$. Then

$$\Delta <\Gamma>= \text{ pure polynomial in } 1/\epsilon$$

without any dependence of dimensionfull parameters. **To simplify calculations we set first zero** *all masses* **and ext. momenta. This would give zero due to IR poles cancelling UV ones (which we want to find). So we take an internal line ℓ with (loop) momentum p and multiply integrand by an m-factor**

$$p^2/(m^2 + p^2)$$

with the result:

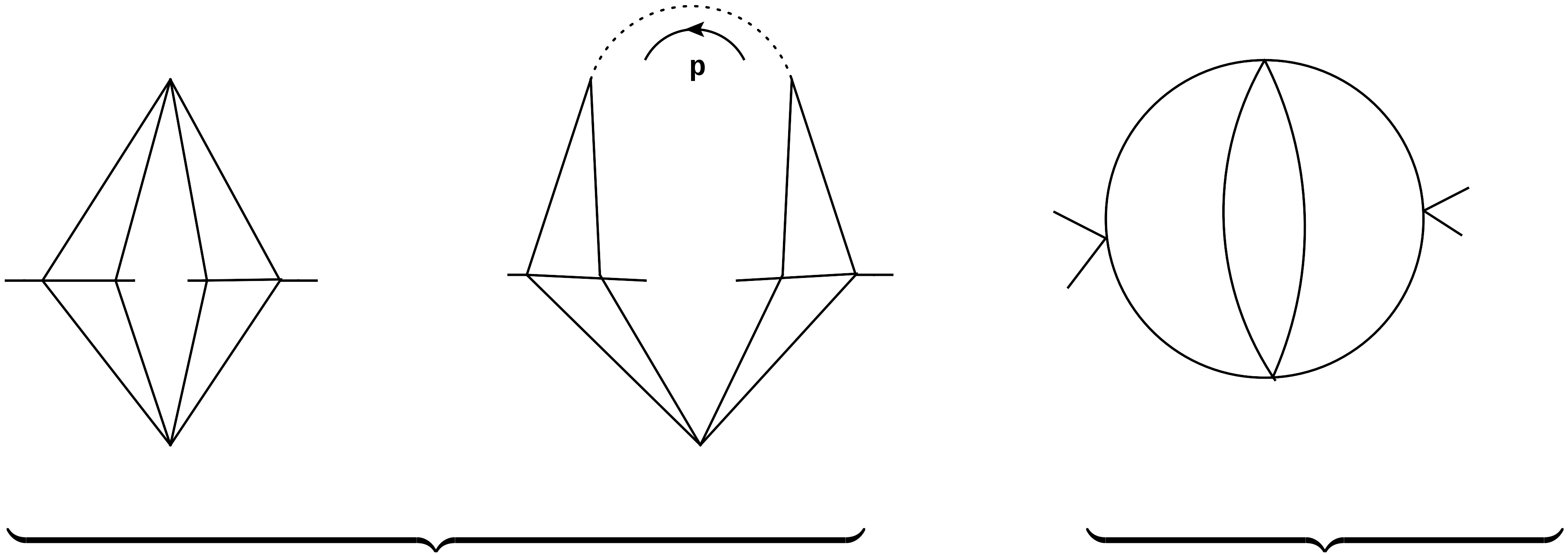
$$<\Gamma>(m) = \int <\Gamma\backslash\,\ell>(p)\,dp\,\frac{1}{(m^2+p^2)}$$

As $<\Gamma\backslash\,\ell>(p) = P(\epsilon)\,(p^2)^{-L\epsilon}$ we need to know $P(\epsilon)$ to $\mathcal{O}(\epsilon^0)$ in order to find the pole part of $<\Gamma>(m)$, which is the most complicated piece of $\Delta <\Gamma>(m) \equiv -K_\epsilon R' <\Gamma>(m)$

The above construction is valid if and only if the IR rearranged amplitude $<\Gamma>(m)$ is free from IR poles!

The complexity of the L-loop p-integral depends on the choice of the line ℓ!

Good choice (if possible!) makes life easier



IRR works

No Chance !!!

Milestones /not all!/ of IRR (+ GPT(x) + IBP + uniqueness)

- ϕ^4 **at four loops** /D. Kazakov, O. Tarasov, A. Vladimirov (1979)/

- **Adler function in QCD at 3 loops** /K. Ch., A. Kataev, ,F. Tkachov (1979)/

- **QCD** β **at 3 loops** /O. Tarasov, A. Vladimirov, A. Zharkov (1979)/

- **Adler function in QCD at 4 loops** /S. Gorishny, A. Kataev, S. Larin, (1991)/

- " Five-loop Konishi in N=4 SYM", B. Eden, P. Heslop, G. Korchemsky, V. Smirnov, E. Sokatchev, (2012)

R^*- and $\tilde{R}$-operations

The requitement of IR-finiteness significanly limits the power of IRR and makes next to impossible its automatization. Solution: to subtract the unwanted IR poles with a new "infrared" $\tilde{R}$-operation /K. Ch., Smirnov (1984), $\ldots$, K. Ch. (1991), D. Batkovich, M. Kompaniets (2014), F. Herzog, B.Ruijl (2017)/

The main statement is that for any (euclidean!) Feynman integral $< \Gamma >$

$$\Delta < \Gamma >= -K_\epsilon \, \tilde{R} \, R' \; < \Gamma >$$

and

$$R^* < \Gamma > \quad \textbf{is finite with } R^* \equiv= \tilde{R} \, R$$

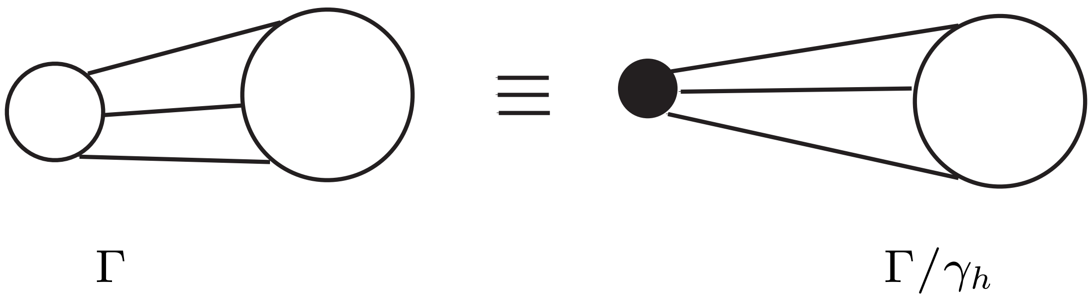
without any requiments IR finiteness

($< \Gamma >$ might be even a massless vacuum graph!)

$\tilde{R}$-operation: some technical details

Let us consider a typical situation: we are given a log-divergent (and Euclidean!) 1PI FI $< \Gamma >$ without external momenta and with only one massive line ℓ_h. How to identify and remove all possible IR poles from it? What is an IR analog of a 1PI UV divergent subgraphs?

Let $\gamma_h \subset \Gamma$ be a hard subgraph of Γ: that is $\ell_h \in \gamma$ and the graph γ/ℓ_h is 1PI. It is usefull to consider two decompositions:



graph decomposition: $\Gamma \equiv \gamma_h \star \Gamma/\gamma_h$ and FI decomposition: $< \Gamma > \equiv < \gamma_h > \star < \Gamma/\gamma_h >$

A *co-subgraph* $\tilde{\gamma}_h$ is defined as Γ/γ_h; the corresponding FI $< \tilde{\gamma}_h >$ is, obviously, 1PI massless tadpole. It is a full IR analog of a UV divergent 1PI subgraph with IR index of divergence $\tilde{\omega}(< \tilde{\gamma}_h >) = -\omega_{UV}(< \tilde{\gamma}_h >)$.

IR c-operator is defined as (assuming that $\tilde{\omega}(< \tilde{\gamma}_h >) = 0$, which is essentially a general case)

$$\tilde{\Delta} < \tilde{\gamma}_h > = \tilde{Z}_{\gamma_h},$$

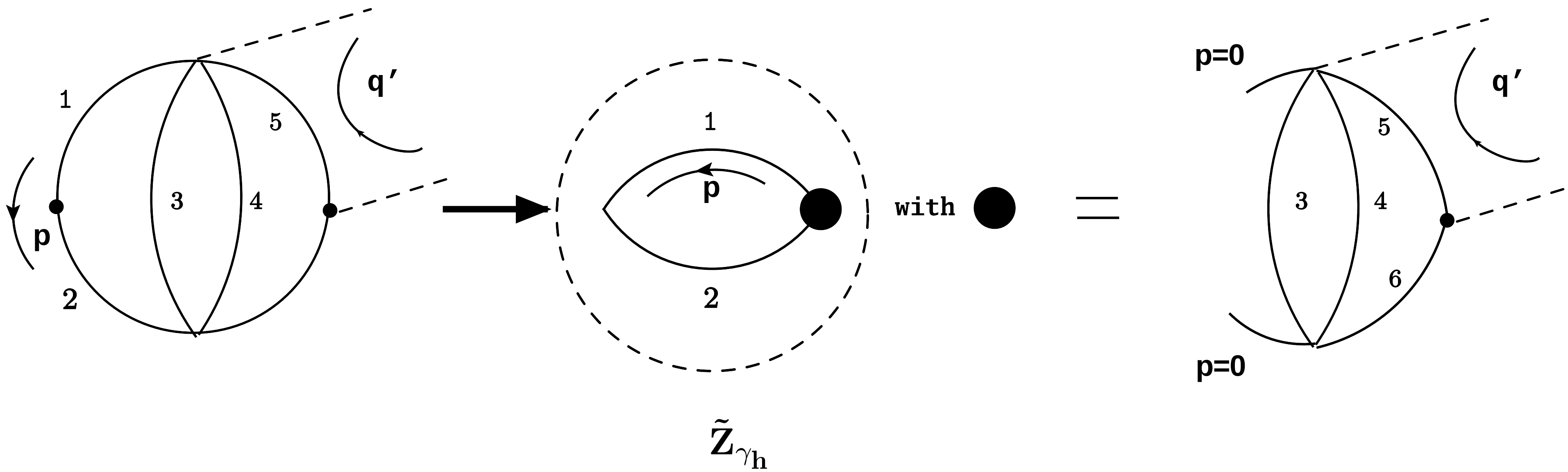
with $\tilde{Z}_{\gamma_h}$ being some pure polynomial in $1/\epsilon$.

Now we define the the IR $\tilde{R}$ operation as follows:

$$\tilde{R} < \Gamma > = < \Gamma > + \sum_{\gamma_h} T_0 < \gamma_h > \tilde{\Delta} < \tilde{\gamma}_h >$$

where the operator T_0 set zero the external momenta of the FI $< \gamma_h >$ which are (soft) *loop* momenta the co-subgraph $\tilde{\gamma}_h$. Our scalar example now looks:

$\Gamma = \{123456\}$ $\tilde{\gamma}_h = 1, 2$ $\gamma_h = \{3456\}$



An important particular case: if $< \Gamma >$ is a massless vacuum graph, then

$$\tilde{R} < \Gamma > = < \Gamma > + \tilde{\Delta} < \Gamma > = \tilde{\Delta} < \Gamma >$$

Comments:

1. the main theorem of $\tilde{R}$-operation is that a proper choice of IR countertems makes $\tilde{R} < \Gamma >$ completely IR finite and, correspondingly, the combination $\tilde{R}R < \Gamma >$ completely finite.

2. For any massless tadpole $< \gamma_0 >$ the IR counterterm $\tilde{\Delta} < \gamma_0 >$ could be found in terms of UV counterterms for $< \gamma_0 >$ and its UV subgraphs.

For instance,

$$R < \tilde{\gamma}_h > \equiv < \tilde{\gamma}_h > + \Delta < \tilde{\gamma}_h > + \tilde{\Delta} < \tilde{\gamma}_h >$$

whence we infer:

$$\tilde{\Delta} < \tilde{\gamma}_h > = -\Delta < \tilde{\gamma}_h > = -\left(-\frac{1}{\epsilon}\right) = +\frac{1}{\epsilon}$$

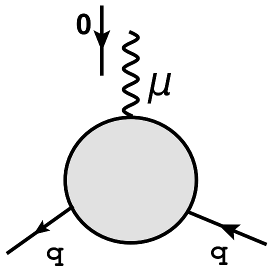
3. main formula for UV Z-factors:

$$Z_\Gamma = -K\tilde{R} \cdot R' < \Gamma > \quad (\star)$$

If $< \Gamma >$ has at least 1 massive line then all IR Z-factors appearing on the rhs of $(\star)$ will come from co-subgraphs with loop number *strictly* less than that of original FI $< \Gamma >$. This nicely explains why we can not start from a purely massless tadpole.

Example of global calculation of an IR Z-factor

Let us consider vertex function $\Gamma_B(a_s^0, q^2) = 1 + \delta\Gamma_B(a_s^0, q^2)$ of a vector quark curent:



$$= \gamma_\mu \cdot \Gamma_B(a_s^0, q^2)$$

Its renormalized (and, hence, *finite*) version reads (note that $Z_V \equiv Z2$, like in QED):

$$\Gamma_R(a_s, q^2) = Z_V\, \Gamma_B(a_s^0, q^2) = Z_V + Z_V\, \delta\Gamma_B(a_s^0, q^2)$$

In fact, the combination $Z_V\, \delta\Gamma_B(a_s^0, q^2) = -K R'\, \Gamma(a_s, q^2)$. Now we set $q \equiv 0$, then the vertex function $\Gamma_R(a_s, q = 0)$ will be *not* zero but Z_V!

Now if we apply, in addition, the IR $\tilde{R}$ operation everything should be finite. Natural normalization of R^* operation is $R^* <$ massless tadpole $>\equiv 0$.
Thus, we have:

$$Z_V + Z_V\, \tilde{\Delta}\, \delta\Gamma_B(a_s^0, q = 0) = 1$$

and, consequently,

$$\tilde{\Delta}\, \delta\Gamma_B(a_s^0, q = 0) = -1 + \frac{1}{Z_V}$$

By defining that $\tilde{\Delta}(1) = 1$ we arrive to a beautiful formula

$$\tilde{\Delta}\,\Gamma_B(a_s^0, q = 0) = \frac{1}{Z_V}$$

Note that the appearance of an inverted Z-factor in the above formula is not by chance: in fact, the $\tilde{R}$ operation is intimately related to so-called R^{-1}-operation (F. Tkachov, G. Pivovarov (1986); K. Ch. (1989)) defined by the equation:

$$R^{-1} \cdot R = 1$$

where R^{-1} is another R-operation with its own c-operator Δ^{-1}.

Given a c-operator Δ , one can always construct corresponding Δ^{-1} one via a recursive procedure. For instance, in 1-loop case

$$\Delta^{-1} < \gamma >= -\Delta < \gamma >$$

For experts in R-operation, forest technique, Hopf algebras and all that

In 1-loop case $\Delta^{-1} \equiv \tilde{\Delta}$. In fact, this is not by chance. One could prove (assuming the natural normalization $R^* <$ massless tadpole $>= 0$ and using purely combinatorial arguments) that the operator

$$\tilde{\Delta} \equiv \Delta^{-1}$$

and that the R^{-1} and Δ^{-1} can be explicitly expressed in terms of Δ as follows*:

$$\Delta^{-1}(\Gamma) = -\Delta(\Gamma) \sum_{F \in F_U\{\Gamma\}} (-)^{|F|} \Delta(F)$$

$$R^{-1}(\Gamma) = \sum_{F \in F_U\{\Gamma\}} (-1)^{|F|} \Delta(F)$$

where F_U stands for the set of all UV *forests* of the graph Γ

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* K. Ch. "Combinatorics OF $\mathbf{R}$-, $\mathbf{R}^{-1}$-, and $\mathbf{R}^*$-Operations and Asymptotic Expansions of Feynman Integrals in the Limit of Large Momenta AND Masses", preprint MPI-Ph/PTh 13/91 (1991); arXiv:1701.08627

For experts in R-operation, forest technique, Hopf algebras and all that

In a sense R^{-1} operation serves a role for minimally renormalized Feynman amplitudes similar to that of *Zimmerman identities* for the momentum subtracted ones. For instance, one could derive, in purely combinatorial way, the following RG equation for a *single* minimally renormalzied FI

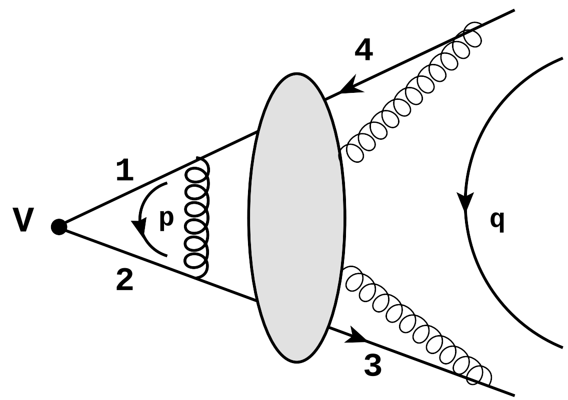
$$-\mu^2 \frac{\partial}{\partial_{\mu^2}} R < \Gamma > \; + \epsilon \cdot L(\Gamma)\, R \, < \Gamma >=$$

$$\sum_{\gamma} L(\gamma)\, \Delta(\gamma)_{\rm rsp} \star \, R \, < \Gamma/\gamma >,$$

where $L(\Gamma)$ is the the loop number of graph Γ and $\Delta(\gamma)_{\rm rsp}$ stand for the *coefficient* in front of the single pole part of the (UV) counterterm $\Delta(\gamma)$

The equation easily leads to the standart RG evolution eqs. for minimally renormalized full Green functions *without* any use of the language of bare charges (Bonnenau, (1980))

Global IR-reduction: a simple example $Z_V = Z_2$



$$R\,\Gamma^V(a_s,q^2) \;=\; Z_V + \,R'\,\delta\Gamma^V(a_s,q^2) = Z_V + Z_V\,R''\,\delta\Gamma^V(a_s,q^2)$$
$$\;=\; Z_V + Z_V\,\delta\,\Gamma^V_B(a_s^0,q^2) = Z_V + Z_V\,\delta\Gamma^V_B(a_s^0,q^2)$$

From finitness we have two ***equivalent*** **formulas to compute** Z_V (the operator K extracts the pole part in ϵ)

$$Z_V = 1 - K\,R'\,\delta\Gamma^V(a_s,q^2) \;\; \textbf{and} \;\; Z_V = 1 - K\,Z_V\,\delta\Gamma^V_B(a_s^0,q^2)$$

Now we introduce an auxiliary mass by multiplying the ***integrand*** **of if every FI from** Γ^V **by a magic m-factor:** $\frac{-p^2}{m^2-p^2}$**. The second eq. (with bare parametrers), STOPS to work, while the first is still operative (due to flexibility of R-operation!).**

Global IR-reduction: a simple example $Z_V = Z_2$

And now we set $q = 0$ and add $\tilde{R}$ to kill unwanted IR poles:

$$Z_V = 1 - K\,\tilde{R}R'\,\delta\Gamma^V(a_s, q = 0, m)$$

Most important now is understand how to write R' in a global form (that is in terms of bare parameters).

The 2 key observations:

(i) UV counterterms which include the V-vertex kill m-factor and transform the diagram with a massive line into a completely massive one;

(ii) Such counterterms are always (by definition) those from Z_V and, thus, R'' contains no such terms at all!

As a result ($Z_V \equiv 1 + \delta Z_V$)

$$\begin{aligned} R'\delta\Gamma^V(a_s, m) &= R''\delta\Gamma^V(a_s^0, m) + \delta Z_V\,\delta\Gamma^V(a_s^0, m = 0) \\ &= \delta\Gamma^V_B(a_s^0, m) + \delta Z_V\,\delta\Gamma^V_B(a_s^0, m = 0) \end{aligned}$$

The application of $\tilde{R}$-operation to remove all IR divergent pieces from the above eq. leads to:

$$\tilde{R}\, R' \delta\Gamma^V(a_s, m) = \delta\Gamma^V_B(a^0_s, m) + \left(\delta\Gamma^V_B(a^0_s, m) \,+\, \delta Z_V\,\right) \cdot \boxed{\left(\frac{1}{Z_V} - 1\right)},$$

where we have boxed the IR Z-factor.

The final closed formula for Z_V reads:

$$Z_V = 1 - K_\epsilon \Big\{ \delta\Gamma^V_B(a^0_s, m)\, \frac{1}{Z_V} - \frac{(Z_V - 1)^2}{Z_V} \Big\}$$

Here the simplicity of the treatment comes from the simple renormalization properties of the V vertex. Similar compact global formulas can be easily derived for the ghost propgator (to find Z^{gh}_2) and for the ghost-ghost gluon vertex.

The really difficults problems are sitting in the gluon slef-nergy (the only way to find Z_3!).

In this case we encounter formally "infinite" UV mixing (for GGT only)

That is in higher orders of PT operators like

$$\bar{\psi} t^a \gamma_\mu A^a_\mu \psi$$

starts to mix with more an more complicated 4-gluon operators like

$$K(a,b,c,d)\,(A^a_\mu A^b_\mu)(A^c_\nu A^b_\nu)$$

with $K(a,b,c,d)$ being a color tensor of rank 4. The # of such color structures increases with # of loops. This leads to enormous problems...

In direct calculations (without IR reduction) we do not see this problem as we always sum over all couplings of 2 external gluons of the gluon self-energy. With IR rearranged diagrams the corresponding cancellations do not exist any more and we should deal with them somehow ...

Theory versus practice: issues with automated IR-reduction

Practical application of the IR-reduction (for not too many diagrams, of course!) is always possible in "manual mode". This is exacly how the first 3- and 4-loop QCD calculations (and even 5-loop ones for the scalar ϕ^4-model) were done in 1979 – 1991.

In general, it is certainly should be possible, though not very simple, to implement the R^*-operation locally (that is diagram-wise). For the scalar theories it was done recently in /D.V. Batkovich, M. Kompaniets (2014)/ and for the more general "tensor" amplitudes appearing in GGT by F. Herzog, B.Ruijl (2017)/

But direct generalization for the GGT encounters *serious* **problems (essentially non-existent for the "massive" way, where the global solution of combinatorics is very simple and well-known!**

Even if we do solve global combinatorics of the IR-reduction and apply R^* locally severe problems still remain.

Conclusions

- **IRR based on R^* operation significantly simplifies RG calculations. It reduces (L+1)-loop RG function in any model to a combination of properly consructed p-integrals; the latter include not only standard UV- but also IR subtractions. It is always possible to do at the level of *separate* diagrams.**
- **IR counterterms are expressible diagramwise (locally or globally) through UV-ones.**
- **The existence of few reduction tools for 4-loop massless propgators (especially) FORCER + known 4-loop masters $\Longrightarrow$ the 5-loop RG functions are *in principle* doable in *any* model.**
- **But: global representation of neccessary IR subtractions (that is on the level of Green functions) strongly depends on the problem and not always easy.**
- **The local R^* approach is more universal and (significantly) more automated than the the gobal one. It requres more computer/time resources and signifucantly less human intervention. But it is not (in its current form) applicable to every RG-problem.**