Yangian symmetry of fishnet graphs

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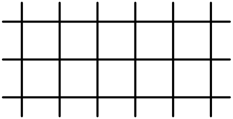
Based on work arXiv:1704.01967 in collaboration with V. Kazakov, F. Loebbert, D. Müller, D. Zhong

Overview

- Planar multi-loop multi-point massless conformal Feynman graphs

Fishnet in 4D  [Zamolodchikov '80]

- Yangian symmetry $\longleftrightarrow$ Integrability
- Higher order symmetry (extends conformal symmetry)
- Set of differential equations
- Integrability of the underlying theory
- 4D biscalar theory (limit of $\mathcal{N} = 4$ SYM) $\longleftrightarrow$ square fishnet lattice
    - Single-trace correlators (off-shell legs)
    - Amplitudes (on-shell legs)
    - Cuts (mixed on-shell/off-shell)

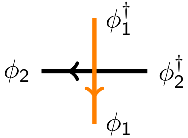
  } Yangian symmetry
- Yangian symmetry is NOT broken by loop corrections
- Generalization: 3D, 6D scalar theory, 4D scalars & fermions

- QFT generating Fishnet graphs (square lattice)
- 4D biscalar theory. Complex scalars ϕ_1, ϕ_2 in adj of $SU(N_c)$

$$\mathcal{L} = \frac{N_c}{2}\,\mathrm{Tr}\left(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2\, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right)$$

[Gurdogan, Kazakov '15]

- A double scaling limit of γ-deformed $\mathcal{N} = 4$ SYM
- "Almost" conformal in the planar limit. $\frac{d\xi}{d\log\mu} = O(N_c^{-2})$
- Integrability (spectrum of anomalous dimensions) [Caetano, Gurdogan, Kazakov '16]
- Non-unitary. Chiral structure



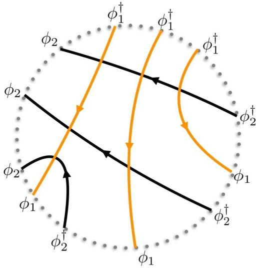
- One Feynman graph per loop order in the planar limit

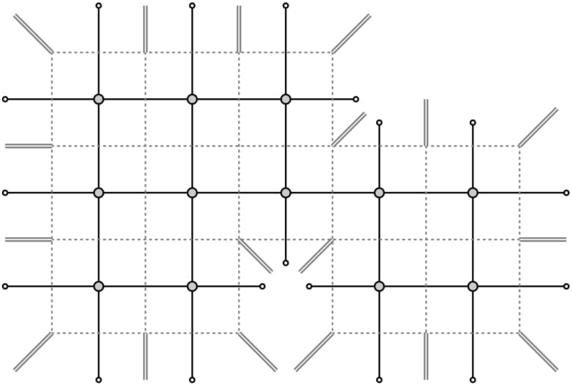
Fishnet graphs with regular boundary

Single-trace correlator

$$\langle \mathrm{Tr}\left[\chi_1(x_1)\,\chi_2(x_2),\ldots\chi_{2M}(x_{2M})\right]\rangle$$

where $\chi_i \in \{\,\phi_1^\dagger, \phi_2^\dagger, \phi_1, \phi_2\}$





Duality transformation

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

x-space – correlator graph

p-space – loop graph with off-shell legs

Yangian of conformal algebra

Conformal algebra $\mathfrak{so}(2,4)$

$$D = -i(x_\mu \partial_\mu + \Delta) \quad , \quad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \,,$$

$$P_\mu = -i\partial_\mu \quad , \quad K_\mu = i(x^2\,\partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu)$$

and its infinite-dimensional extension – Yangian [Drinfeld '85]

$$J \;,\; \widehat{J} \;,\; [\widehat{J},\widehat{J}] \;,\; [\widehat{J},[\widehat{J},\widehat{J}]] \;,\; [[\widehat{J},\widehat{J}],[\widehat{J},\widehat{J}]] \;\; \ldots$$

where level-zero generators $J^A \in \mathfrak{so}(2,4)$ and level-one generators $\widehat{J}$ satisfy

$$\left[J^A, J^B\right] = f^{AB}{}_C\, J^C \;,\; \left[J^A, \widehat{J}^B\right] = f^{AB}{}_C\, \widehat{J}^C \;,$$

Jacobi and Serre relations – cubic in J, $\widehat{J}$

Evaluation representation with evaluation parameters v_k

$$\widehat{J}^A = \tfrac{1}{2} f^A{}_{BC} \sum_{j<k} J_j^C J_k^B + \sum_k v_k J_k^A$$

Yangian symmetry of the Fishnet graphs

$$J^A |\text{Fishnet}\rangle = \widehat{J}^A |\text{Fishnet}\rangle = 0$$

Yangian of conformal algebra

Conformal algebra $\mathfrak{so}(2,4)$

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and its infinite-dimensional extension – Yangian [Drinfeld '85]

$$J \ , \ \widehat{J} \ , \ [\widehat{J}, \widehat{J}] \ , \ [\widehat{J}, [\widehat{J}, \widehat{J}]] \ , \ [[\widehat{J}, \widehat{J}], [\widehat{J}, \widehat{J}]] \ \ \ldots$$

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Jacobi and Serre relations – cubic in J, $\widehat{J}$

Evaluation representation with evaluation parameters v_k

$$\widehat{P}^\mu = -\tfrac{i}{2} \sum_{j<k} \left[(L_j^{\mu\nu} + \eta^{\mu\nu} D_j) P_{k,\nu} - (j \leftrightarrow k)\right] + \sum_k v_k P_k^\mu$$

Yangian symmetry of the Fishnet graphs

$$J^A |\text{Fishnet}\rangle = \widehat{J}^A |\text{Fishnet}\rangle = 0$$

Lax and monodomry matrix

- Lax matrix with the spectral parameter u

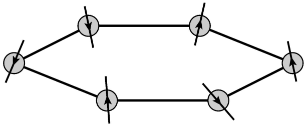
$$L(u) = \begin{pmatrix} 1+\frac{1}{u}\mathcal{J}_{11} & \frac{1}{u}\mathcal{J}_{12} & \frac{1}{u}\mathcal{J}_{13} & \frac{1}{u}\mathcal{J}_{14} \\ \frac{1}{u}\mathcal{J}_{21} & 1+\frac{1}{u}\mathcal{J}_{22} & \frac{1}{u}\mathcal{J}_{23} & \frac{1}{u}\mathcal{J}_{24} \\ \frac{1}{u}\mathcal{J}_{31} & \frac{1}{u}\mathcal{J}_{32} & 1+\frac{1}{u}\mathcal{J}_{33} & \frac{1}{u}\mathcal{J}_{34} \\ \frac{1}{u}\mathcal{J}_{41} & \frac{1}{u}\mathcal{J}_{42} & \frac{1}{u}\mathcal{J}_{43} & 1+\frac{1}{u}\mathcal{J}_{44} \end{pmatrix}$$

consists of $\mathfrak{so}(2,4)$ generators $\mathcal{J}_{ij} \in \mathrm{span}\{D, P_\mu, K_\nu, L_{\mu\nu}\}$

- n-point monodromy matrix with inhomogeneities $\delta_1, \delta_2, \ldots, \delta_n$

$$T(u;\vec{\delta}) = L_n(u+\delta_n)\ldots L_2(u+\delta_2)\, L_1(u+\delta_1)$$

$$T_{ab}(u;\vec{\delta}) = \delta_{ab} + \sum_{k\geq 0} u^{-1-k}\mathcal{J}^{(k)}_{ab}$$



Quantum spin chain with noncompact representations of $\mathfrak{so}(2,4)$

Lax and monodomry matrix

RTT-relation defines the quadratic algebra for $\{\mathcal{J}^{(k)}\}$

$$R_{ae,bf}(u-v)\ T_{ec}(u)\ T_{fd}(v) = T_{ae}(v)\ T_{bf}(u)\ R_{ec,fd}(u-v)$$

[Faddeev, Kulish, Sklyanin, Takhtajan,...'79]

with Yang's R-matrix $R_{ab,cd}(u) = \delta_{ab}\delta_{cd} + u\delta_{ad}\delta_{bc}$.
RTT is compatible with the co-product.

Yangian symmetry $\longleftrightarrow$ **Eigenvalue relation for the monodromy matrix**

$$L_n(u+\delta_n)\dots L_2(u+\delta_2)\ L_1(u+\delta_1)\ |\text{Fishnet}\rangle = \lambda(u;\vec{\delta})\ |\text{Fishnet}\rangle \cdot \mathbf{1}$$

and expanding this matrix equation in the spectral parameter u,

$$\mathcal{J}^{(k)}_{ab}|\text{Fishnet}\rangle = \lambda_n(\vec{\delta})\delta_{ab}\ |\text{Fishnet}\rangle$$

For oscillator representations [D.C., Kirschner '13; D.C., Kirschner, Derkachov '13; Frassek, Kanning, Ko, Staudacher '13; Broedel, de Leeuw, Rosso '14; . . .]

Conformal Lax

Lax matrix depends on parameters $(u, \Delta) \Leftrightarrow (u_+, u_-)$

$$L(u_+, u_-) = \begin{pmatrix} u_+ \cdot \mathbf{1} - \mathbf{p} \cdot \mathbf{x} & \mathbf{p} \\ -\mathbf{x} \cdot \mathbf{p} \cdot \mathbf{x} + (u_+ - u_-) \cdot \mathbf{x} & u_- \cdot \mathbf{1} + \mathbf{x} \cdot \mathbf{p} \end{pmatrix}$$

where [D.C., Derkachov, Isaev '12]

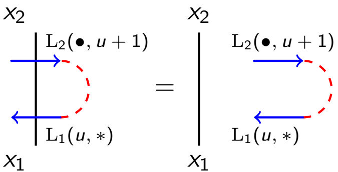
$$\mathbf{x} = -i\overline{\sigma}_\mu x^\mu \ , \ \ \mathbf{p} = -\tfrac{i}{2}\sigma^\mu \partial_\mu \ , \ \ u_+ = u + \tfrac{\Delta - 4}{2} \ , \ \ u_- = u - \tfrac{\Delta}{2}$$

- Local vacuum of the Lax

$$L(u, u+2) \cdot 1 = (u+2) \cdot \mathbf{1}$$

- Intertwining relation with the scalar propagator

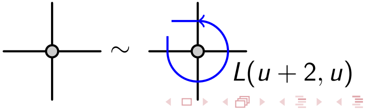
$x_{i,j} \equiv x_i - x_j$



$$L_1(u+1, *) L_2(\bullet, u)\ x_{12}^{-2}$$

$$= x_{12}^{-2}\ L_1(u, *) L_2(\bullet, u+1)$$

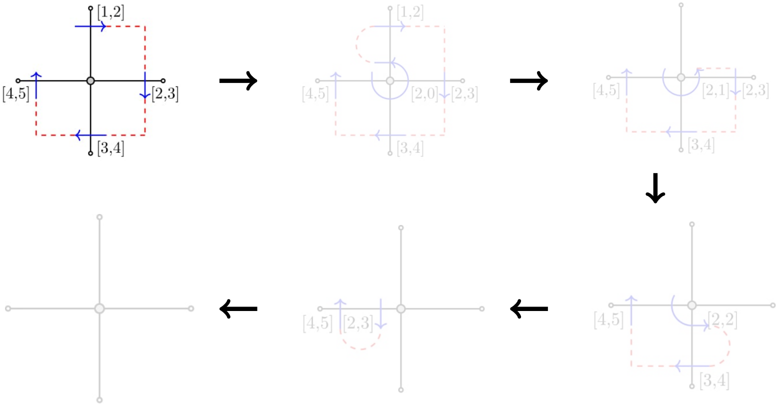
- Integration by parts



$$L^T(u+2, u) \cdot 1 = (u+2) \cdot \mathbf{1}$$

$[i,j] \equiv L(u+i, u+j)$

Example: Cross integral



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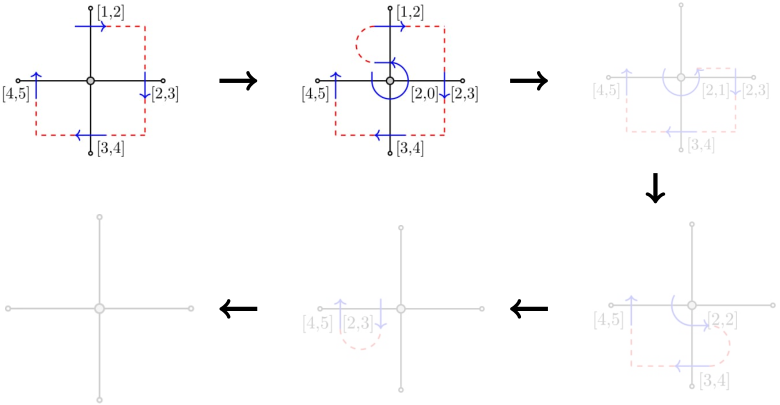

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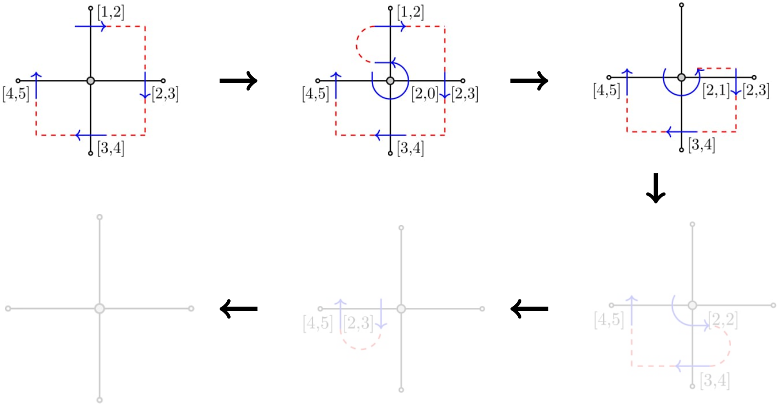

$$L^T(u+2,u)\cdot 1 = (u+2)\cdot \mathbf{1} \qquad , \qquad L(u,u+2)\cdot 1 = (u+2)\cdot \mathbf{1}$$

$$x_{12}^{-2} L_1(u,*) L_2(\bullet, u+1) = L_1(u+1,*) L_2(\bullet, u) x_{12}^{-2}$$

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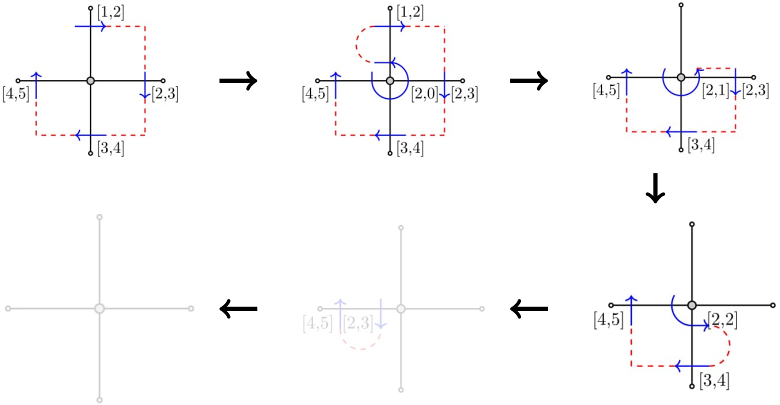
Example: Cross integral



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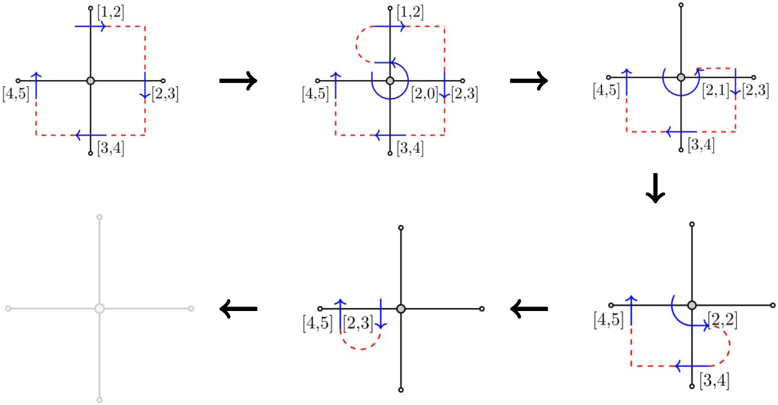

$$L^T(u+2,u)\cdot 1 = (u+2)\cdot \mathbf{1} \qquad , \qquad L(u,u+2)\cdot 1 = (u+2)\cdot \mathbf{1}$$

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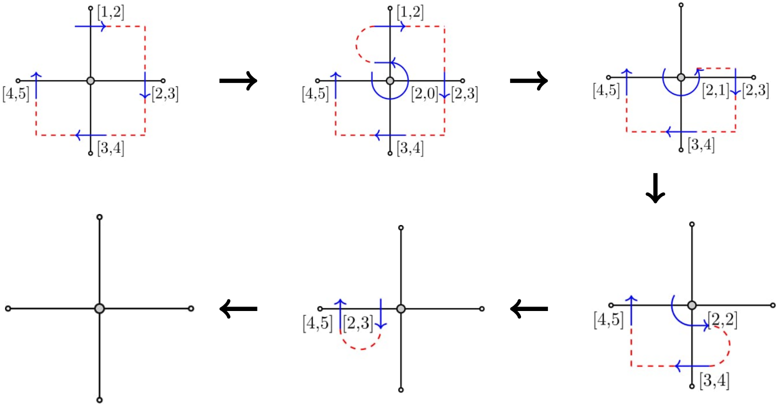


$L^T(u+2, u) \cdot 1 = (u+2) \cdot \mathbf{1}$, $L(u, u+2) \cdot 1 = (u+2) \cdot \mathbf{1}$

$x_{12}^{-2} L_1(u, *) L_2(\bullet, u+1) = L_1(u+1, *) L_2(\bullet, u) x_{12}^{-2}$

Example: Cross integral

Yangian symmetry of the cross integral

$$L_4[4,5]L_3[3,4]L_2[2,3]L_1[1,2]\,|\text{cross}\rangle = [3][4]^2[5]\cdot|\text{cross}\rangle\cdot\mathbf{1}$$

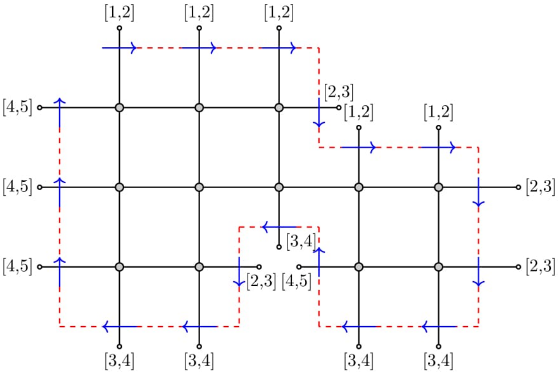
where $[i,j] \equiv L(u+i,u+j)$ and $[i] \equiv u+i$

$$|\text{cross}\rangle = \int d^4x_0 \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} = x_{13}^{-2} x_{24}^{-2}\Phi(s,t)$$

Conformal cross-ratios s,t. The Yangian symmetry implies DE

$$\Phi + (3s-1)\frac{\partial\Phi}{\partial s} + 3t\frac{\partial\Phi}{\partial t} + (s-1)s\frac{\partial^2\Phi}{\partial s^2} + t^2\frac{\partial^2\Phi}{\partial t^2} + 2st\frac{\partial^2\Phi}{\partial s\partial t} = 0$$

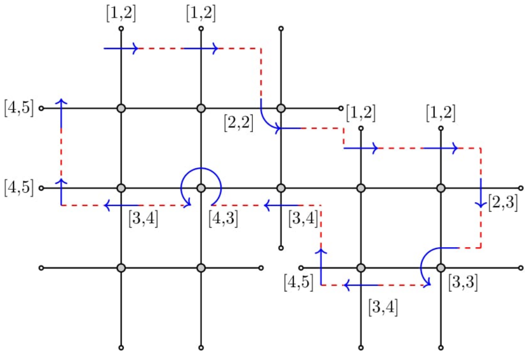
Fishnet graphs with regular boundary



$$\left(\prod_{i\in\mathcal{C}} L_i[\delta_i^+,\delta_i^-]\right)|\text{Fishnet}\rangle = \left(\prod_{i\in\mathcal{C}_{\text{out}}}[\delta_i^+][\delta_i^-]\right)|\text{Fishnet}\rangle\cdot\mathbf{1}$$

where $\mathcal{C}=\mathcal{C}_{\text{in}}\cup\mathcal{C}_{\text{out}}$ is the boundary of the graph

Fishnet graphs with regular boundary



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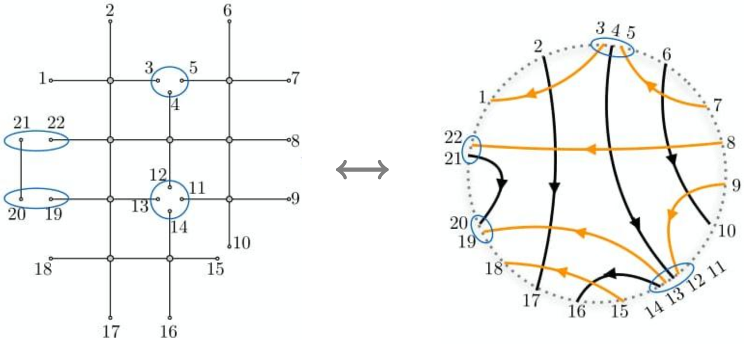
Fishnet graphs with irregular boundary

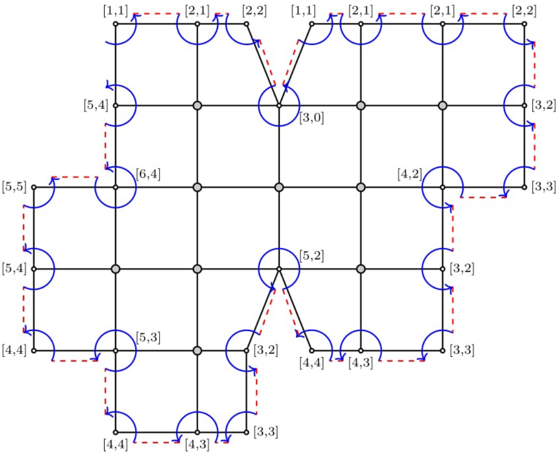


Single-trace correlator

$$\langle \mathrm{Tr}\left[\chi_1(x_1)\,\chi_2(x_2),\ldots\chi_{2M}(x_{2M})\right]\rangle$$

where $\chi_i \in \{\,\phi_1^\dagger, \phi_2^\dagger, \phi_1, \phi_2\}$ and some of x_i's are identified

Fishnet graphs with irregular boundary

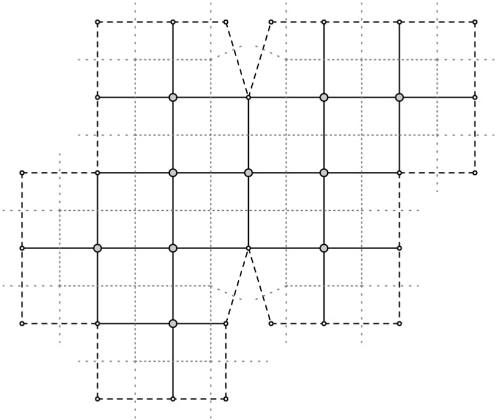


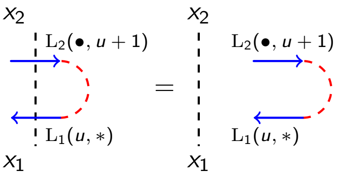
$$\left(\prod_{i\in\mathcal{C}} L_i[\delta_i^+,\delta_i^-]\right)|\text{Fishnet}\rangle = \lambda(u;\vec{\delta})\,|\text{Fishnet}\rangle\cdot\mathbf{1}$$

and eigenvalue λ is fixed by algebraic equations (implied by cyclic symmetry).

Scattering amplitudes and cuts of fishnet graphs

Duality transformation $x_i^\mu - x_{i+1}^\mu = p_i^\mu$. Amplitude $p_i^2 = 0 \Leftrightarrow x_{i,i+1}^2 = 0$





$$L_1(u+1,*)L_2(\bullet,u)\ \delta(x_{12}^2)$$

$$= \delta(x_{12}^2)\ L_1(u,*)L_2(\bullet,u+1)$$

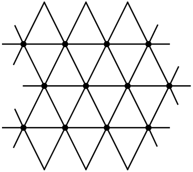
$$T(u)\,|\text{Fishnet}\rangle = \lambda(u)\,|\text{Fishnet}\rangle\cdot\mathbf{1} \quad\Rightarrow\quad T(u)\,|\text{Fishnet}\rangle_{\text{cut}} = \lambda(u)\,|\text{Fishnet}\rangle_{\text{cut}}\cdot\mathbf{1}$$

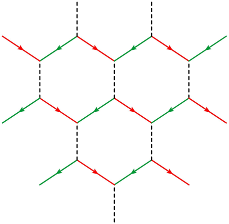
for any cut $x_{ij}^{-2} \to \delta(x_{ij}^2)$.

Generalizations: tri-scalar theories in 3D, 6D, and scalar-fermion theories in 4D

$$\mathcal{L}_{3D,\mathrm{int}} = \xi N_c \mathrm{Tr}\, Y_1 Y_4^\dagger Y_2 Y_1^\dagger Y_4 Y_2^\dagger$$

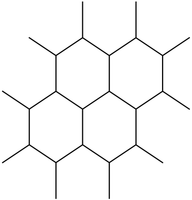
[Caetano, Gurdogan, Kazakov '16]





$$\mathcal{L}_{6D,\mathrm{int}} = N_c \mathrm{Tr}\left(\xi_1\, \phi_1^\dagger \phi_2 \phi_3 + \xi_2\, \phi_1 \phi_2^\dagger \phi_3^\dagger\right)$$

[Mamroud, Torrents '17]



$$\mathcal{L}_{4D,\mathrm{int}} = \mathcal{L}_{\mathsf{Yukawa}} + \mathcal{L}_{\phi^4}$$

scalar and fermion fields

[Gurdogan, Kazakov '15]

Conclusions

- Conformal Yangian symmetry of single-trace correlators and amplitudes
- RTT-realization of the Yangian algebra
- Propagators – intertwining operators of the Laxes and monodromy
- Yangian invariance is a consequence of (bi)local operator relations
- Lasso proof of the Yangian invariance
- The approach also works for Fishnet graphs in 3D, 6D, and for scalar-fermion theories in 4D