Yangian symmetry of fishnet graphs

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Overview

- Planar multi-loop multi-point massless conformal Feynman graphs

- Yangian symmetry
- Higher order symmetry (extends conformal symmetry)
- Set of differential equations
- Integrability of the underlying theory
- 4D biscalar theory (limit of $\mathcal{N} = 4$ SYM) ↔ square fishnet lattice
  - Single-trace correlators (off-shell legs)
  - Amplitudes (on-shell legs)
  - Cuts (mixed on-shell/off-shell)
- Yangian symmetry is NOT broken by loop corrections
- Generalization: 3D, 6D scalar theory, 4D scalars & fermions

Fishnet in 4D

[Zamolodchikov '80]
• QFT generating Fishnet graphs (square lattice)

• 4D biscalar theory. Complex scalars $\phi_1, \phi_2$ in adj of $SU(N_c)$

$$\mathcal{L} = \frac{N_c}{2} \text{Tr} \left( \partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right)$$

[Gurdogan, Kazakov ’15]

• A double scaling limit of $\gamma$-deformed $\mathcal{N} = 4$ SYM

• ”Almost” conformal in the planar limit. $\frac{d\xi}{d \log \mu} = O(N_c^{-2})$

• Integrability (spectrum of anomalous dimensions) [Caetano, Gurdogan, Kazakov ’16]

• Non-unitary. Chiral structure

• One Feynman graph per loop order in the planar limit
Fishnet graphs with regular boundary

Single-trace correlator

$$\langle \text{Tr} \left[ \chi_1(x_1) \chi_2(x_2) \ldots \chi_{2M}(x_{2M}) \right] \rangle$$

where $$\chi_i \in \{ \phi_1^\dagger, \phi_2^\dagger, \phi_1, \phi_2 \}$$

Duality transformation

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

$$x$$-space – correlator graph

$$p$$-space – loop graph with off-shell legs
Yangian of conformal algebra

Conformal algebra $\mathfrak{so}(2,4)$

\[ D = -i(x_\mu \partial_\mu + \Delta) \]
\[ L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \]
\[ P_\mu = -i\partial_\mu \]
\[ K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu) \]

and its infinite-dimensional extension – Yangian

\[ J, \hat{J}, [J, \hat{J}], [J, [\hat{J}, \hat{J}]], [[\hat{J}, \hat{J}], [\hat{J}, \hat{J}]], \ldots \]

where level-zero generators $J^A \in \mathfrak{so}(2,4)$ and level-one generators $\hat{J}$ satisfy

\[ [J^A, J^B] = f^{AB}_C J^C \]
\[ [J^A, \hat{J}^B] = f^{AB}_C \hat{J}^C \]

Jacobi and Serre relations – cubic in $J, \hat{J}$

Evaluation representation with evaluation parameters $v_k$

\[ \hat{J}^A = \frac{1}{2} f^{AB}_C \sum_{j<k} J^C_j J^B_k + \sum_k v_k J^A_k \]

Yangian symmetry of the Fishnet graphs

\[ J^A |\text{Fishnet}\rangle = \hat{J}^A |\text{Fishnet}\rangle = 0 \]
Yangian of conformal algebra

Conformal algebra $\mathfrak{so}(2,4)$

\[ D = -i(x^\mu \partial_\mu + \Delta) \, , \, L_{\mu\nu} = i(x^\mu \partial_\nu - x^\nu \partial_\mu) \, , \]
\[ P_\mu = -i\partial_\mu \, , \, K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x^\mu) \]

and its infinite-dimensional extension – Yangian

\[ J \, , \, \hat{J} \, , \, [\hat{J}, \hat{J}] \, , \, [\hat{J}, [\hat{J}, \hat{J}]] \, , \, [[[\hat{J}, \hat{J}], [\hat{J}, \hat{J}]]] \ldots \]

where level-zero generators $J^A \in \mathfrak{so}(2,4)$ and level-one generators $\hat{J}$ satisfy

\[ [J^A, J^B] = f^{AB}_C J^C \, , \, [J^A, \hat{J}^B] = f^{AB}_C \hat{J}^C \, , \, \text{Jacobi and Serre relations – cubic in } J, \hat{J} \]

Evaluation representation with evaluation parameters $v_k$

\[ \hat{P}^\mu = -\frac{i}{2} \sum_{j<k} \left[ (L^\mu_{j\nu} + \eta^\mu\nu D_j) P_{k,\nu} - (j \leftrightarrow k) \right] + \sum_k v_k P^\mu_k \]

Yangian symmetry of the Fishnet graphs

\[ J^A |\text{Fishnet}\rangle = \hat{J}^A |\text{Fishnet}\rangle = 0 \]
Lax and monodromy matrix

- Lax matrix with the spectral parameter $u$

$$L(u) = \begin{pmatrix} 1 + \frac{1}{u} J_{11} & \frac{1}{u} J_{12} & \frac{1}{u} J_{13} & \frac{1}{u} J_{14} \\ \frac{1}{u} J_{21} & 1 + \frac{1}{u} J_{22} & \frac{1}{u} J_{23} & \frac{1}{u} J_{24} \\ \frac{1}{u} J_{31} & \frac{1}{u} J_{32} & 1 + \frac{1}{u} J_{33} & \frac{1}{u} J_{34} \\ \frac{1}{u} J_{41} & \frac{1}{u} J_{42} & \frac{1}{u} J_{43} & 1 + \frac{1}{u} J_{44} \end{pmatrix}$$

- consists of $\mathfrak{so}(2, 4)$ generators $J_{ij} \in \text{span}\{D, P_\mu, K_\nu, L_{\mu\nu}\}$

- n-point monodromy matrix with inhomogeneities $\delta_1, \delta_2, \ldots, \delta_n$

$$T(u; \vec{\delta}) = L_n(u + \delta_n) \cdots L_2(u + \delta_2) L_1(u + \delta_1)$$

$$T_{ab}(u; \vec{\delta}) = \delta_{ab} + \sum_{k \geq 0} u^{-1-k} J_{ab}^{(k)}$$

Quantum spin chain with noncompact representations of $\mathfrak{so}(2, 4)$
Lax and monodromy matrix

RTT-relation defines the quadratic algebra for \( \{ J^{(k)} \} \)

\[
R_{ae,bf}(u-v) \ T_{ec}(u) \ T_{fd}(v) = T_{ae}(v) \ T_{bf}(u) \ R_{ec,fd}(u-v)
\]

[Faddeev, Kulish, Sklyanin, Takhtajan,...'79]

with Yang’s R-matrix \( R_{ab,cd}(u) = \delta_{ab} \delta_{cd} + u \delta_{ad} \delta_{bc} \).

RTT is compatible with the co-product.

Yangian symmetry \( \leftrightarrow \) Eigenvalue relation for the monodromy matrix

\[
L_n(u + \delta_n) \ldots L_2(u + \delta_2) \ L_1(u + \delta_1) \ |\text{Fishnet}\rangle = \lambda(u; \vec{\delta}) \ |\text{Fishnet}\rangle \cdot 1
\]

and expanding this matrix equation in the spectral parameter \( u \),

\[
J^{(k)}_{ab} \ |\text{Fishnet}\rangle = \lambda_n(\vec{\delta}) \delta_{ab} \ |\text{Fishnet}\rangle
\]

For oscillator representations [D.C., Kirschner '13; D.C., Kirschner, Derkachov '13; Frassek, Kanning, Ko, Staudacher '13; Broedel, de Leeuw, Rosso '14; ...]
Conformal Lax

Lax matrix depends on parameters \((u, \Delta) \Leftrightarrow (u_+, u_-)\)

\[
L(u_+, u_-) = \begin{pmatrix}
u_+ \cdot 1 - p \cdot x & p \\
-x \cdot p \cdot x + (u_+ - u_-) \cdot x & u_- \cdot 1 + x \cdot p
\end{pmatrix}
\]

where

\[
x = -i \sigma_\mu x^\mu, \quad p = -i \frac{1}{2} \sigma^\mu \partial_\mu, \quad u_+ = u + \frac{\Delta - 4}{2}, \quad u_- = u - \frac{\Delta}{2}
\]

- **Local vacuum of the Lax**

\[
L(u, u + 2) \cdot 1 = (u + 2) \cdot 1
\]

- **Intertwining relation with the scalar propagator**

\[
L_1(u + 1, \star) L_2(\bullet, u) x_{12}^{-2}
\]

\[
= x_{12}^{-2} L_1(u, \star) L_2(\bullet, u + 1)
\]

- **Integration by parts**

\[
L^T(u + 2, u) \cdot 1 = (u + 2) \cdot 1
\]
Example: Cross integral

\[ [i,j] \equiv L(u+i, u+j) \]
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Example: Cross integral

\[ [i,j] \equiv L(u+i, u+j) \]

\[
L^T(u + 2, u) \cdot 1 = (u + 2) \cdot 1, \quad L(u, u + 2) \cdot 1 = (u + 2) \cdot 1
\]

\[
x_{12}^{-2} L_1(u, *) L_2(\bullet, u + 1) = L_1(u + 1, *) L_2(\bullet, u) x_{12}^{-2}
\]
Example: Cross integral

\[ [i,j] \equiv L(u+i, u+j) \]

\[ L^T(u+2, u) \cdot 1 = (u+2) \cdot 1 \]
\[ L(u, u+2) \cdot 1 = (u+2) \cdot 1 \]

\[ x_{12}^{-2} L_1(u, *) L_2(\bullet, u+1) = L_1(u+1, *) L_2(\bullet, u) x_{12}^{-2} \]
Example: Cross integral

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[i,j] \equiv L(u+i, u+j)
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L^T(u+2, u) \cdot 1 = (u + 2) \cdot 1, \quad L(u, u+2) \cdot 1 = (u + 2) \cdot 1
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x_{12}^{-2} L_1(u, *) L_2(\bullet, u+1) = L_1(u+1, *) L_2(\bullet, u) x_{12}^{-2}
\]
Example: Cross integral

\[ [i,j] \equiv L(u+i, u+j) \]

\[
L^T(u + 2, u) \cdot 1 = (u + 2) \cdot 1, \quad L(u, u + 2) \cdot 1 = (u + 2) \cdot 1
\]

\[
x_{12}^{-2} L_1(u, \ast) L_2(\bullet, u + 1) = L_1(u + 1, \ast)L_2(\bullet, u)x_{12}^{-2}
\]
Example: Cross integral

Yangian symmetry of the cross integral

\[ L_4[4, 5]L_3[3, 4]L_2[2, 3]L_1[1, 2]|\text{cross}\rangle = [3][4]^2[5] \cdot |\text{cross}\rangle \cdot 1 \]

where \([i, j] \equiv L(u + i, u + j)\) and \([i] \equiv u + i\)

\[ |\text{cross}\rangle = \int d^4x_0 \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2} = x_{13}^{-2} x_{24}^{-2} \Phi(s, t) \]

Conformal cross-ratios \(s, t\). The Yangian symmetry implies DE

\[ \Phi + (3s - 1) \frac{\partial \Phi}{\partial s} + 3t \frac{\partial \Phi}{\partial t} + (s - 1)s \frac{\partial^2 \Phi}{\partial s^2} + t^2 \frac{\partial^2 \Phi}{\partial t^2} + 2st \frac{\partial^2 \Phi}{\partial s \partial t} = 0 \]
Fishnet graphs with regular boundary

\[
\left( \prod_{i \in C} L_i [\delta_i^+, \delta_i^-] \right) |\text{Fishnet}\rangle = \left( \prod_{i \in C_{\text{out}}} [\delta_i^+] [\delta_i^-] \right) |\text{Fishnet}\rangle \cdot 1
\]

where \( C = C_{\text{in}} \cup C_{\text{out}} \) is the boundary of the graph.
Fishnet graphs with regular boundary

\[
\left( \prod_{i \in C} L_i \left[ \delta_i^+, \delta_i^- \right] \right) |\text{Fishnet}\rangle = \left( \prod_{i \in C_{\text{out}}} \left[ \delta_i^+ \right] \left[ \delta_i^- \right] \right) |\text{Fishnet}\rangle \cdot 1
\]

where \( C = C_{\text{in}} \cup C_{\text{out}} \) is the boundary of the graph
Fishnet graphs with irregular boundary

Single-trace correlator

$$\left\langle \text{Tr} \left[ \chi_1(x_1) \chi_2(x_2), \ldots \chi_{2M}(x_{2M}) \right] \right\rangle$$

where $$\chi_i \in \{ \phi_1^\dagger, \phi_2^\dagger, \phi_1, \phi_2 \}$$ and some of $$x_i$$'s are identified
Fishnet graphs with irregular boundary

\[
\left( \prod_{i \in \mathcal{C}} L_i[\delta_i^+, \delta_i^-] \right) |\text{Fishnet}\rangle = \lambda(u; \vec{\delta}) |\text{Fishnet}\rangle \cdot \mathbf{1}
\]

and eigenvalue $\lambda$ is fixed by algebraic equations (implied by cyclic symmetry).
Scattering amplitudes and cuts of fishnet graphs

Duality transformation $x_i^\mu - x_{i+1}^\mu = p_i^\mu$. Amplitude $p_i^2 = 0 \iff x_{i,i+1}^2 = 0$

\[
T(u) |\text{Fishnet}\rangle = \lambda(u) |\text{Fishnet}\rangle \cdot 1 \quad \Rightarrow \quad T(u) |\text{Fishnet}\rangle_{\text{cut}} = \lambda(u) |\text{Fishnet}\rangle_{\text{cut}} \cdot 1
\]

for any cut $x_{ij}^{-2} \rightarrow \delta(x_{ij}^2)$. 
Generalizations: tri-scalar theories in 3D, 6D, and scalar-fermion theories in 4D

\[ \mathcal{L}_{3D,\text{int}} = \xi N_c \text{Tr} \ Y_1 Y_4^\dagger Y_2 Y_1^\dagger Y_4 Y_2^\dagger \]

[Caetano, Gurdogan, Kazakov '16]

\[ \mathcal{L}_{4D,\text{int}} = \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\phi^4} \]

scalar and fermion fields

[Caetano, Gurdogan, Kazakov '16]

\[ \mathcal{L}_{6D,\text{int}} = N_c \text{Tr} \left( \xi_1 \phi_1^\dagger \phi_2 \phi_3 + \xi_2 \phi_1 \phi_2^\dagger \phi_3^\dagger \right) \]

[Mamroud, Torrents '17]
Conclusions

- Conformal Yangian symmetry of single-trace correlators and amplitudes
- RTT-realization of the Yangian algebra
- Propagators – intertwining operators of the Laxes and monodromy
- Yangian invariance is a consequence of (bi)local operator relations
- Lasso proof of the Yangian invariance
- The approach also works for Fishnet graphs in 3D, 6D, and for scalar-fermion theories in 4D