Connecting quantum field theories across the dimensions via large N

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Background

Recent activity in quantum field theories has gone beyond our usual spacetime dimension; both above and below four dimensions

Problems in condensed matter theory, such as graphene physics, informed by two and three dimensional field theories

Above four dimensions higher spin AdS/CFT's are related through dualites to six dimensional scalar cubic theories [Polyakov & Klebanov]

Present main activity is in understanding conformal field theories beyond two dimensions; *a*-theorem, (nonperturbative) fixed points, critical exponents, relevant operators, field theories in the same universality class, ultraviolet-infrared duality across dimensions, conformal windows, asymptotic safety

Focus here will be on recent work in scalar field theories and also in gauge theories

Renormalization group functions of renormalizable coninuum quantum field theories are a key to understanding underlying critical point properties

Fixed points are defined as the zeroes of the β-function; clearly free field theory is the trivial fixed point

Value of the renormalization group functions at a fixed point correspond to critical exponents - these are renormalization group invariants and in principle can be measured physically

Various methods used to estimate exponents theoretically:

- ϵ expansion in $d = D - 2\epsilon$ dimensions where D is critical dimension
- direct evaluation in fixed (odd) dimensions
- strong coupling
- lattice or numerical evaluation
- large N techniques
- conformal bootstrap [Rychkov et al]

Renormalization group functions are being determined to very high accuracy by a range of perturbative methods and accord with experimental measurements

Resummation of renormalization group functions for five (six) loop four dimensional ϕ^4 theory in $d = 4 - 2\epsilon$ dimensions are competitive with fixed dimension six loop evaluation

In d-dimensions the relevant fixed point is the Wilson-Fisher (WF) fixed point

There are a large number of higher dimensional theories in the same universality class of the $O(N)$ ϕ^4 at the WF fixed point which leads to ultraviolet completion

Idea of the connection of ultraviolet stable fixed point in higher dimensional theory with infrared stable fixed point in lower dimensional theory has been around for many years [McKane]

Developments post two dimensional conformal field theory revolution seek to use conformal bootstrap and related methods to exploit conformal symmetry structures in d-dimensions

Critical properties such as exponents and finding the spectrum of CFT's is important

ϕ^4 theory and the large N expansion

Four dimensional ϕ^4 scalar field theory with $O(N)$ symmetry

$$L \;=\; \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \;-\; \frac{g^2}{8}\left(\phi^i\phi^i\right)^2$$

It is in the same universality class as the $O(N)$ nonlinear σ model at the Wilson-Fisher fixed point

$$L^\sigma \;=\; \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \;+\; \frac{1}{2}\sigma\left(\phi^i\phi^i - \frac{1}{\lambda}\right)$$

At the Lagrangian level $O(N)$ ϕ^4 theory can be rewritten as

$$L \;=\; \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i \;+\; \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{2g}$$

Both Lagrangians have the same basic interaction; $[\phi^i] = d/2 - 1$ and $[\sigma] = 2$

In terms of perturbative renormalizability L^σ is only perturbatively renormalizable in strictly two dimensions; beyond two it is perturbatively non-renormalizable

By contrast ϕ^4 theory is perturbatively renormalizable in four dimensions and superrenormalizable below four

For ϕ^4 theory first few terms of d-dimensional $\overline{\text{MS}}$ β-function are

$$\begin{aligned}\beta(g) &= (d-4)\frac{g}{2} + [N+8]\frac{g^2}{6} - [3N+14]\frac{g^3}{6} \\ &+ \left[33N^2 + 922N + 2960 + 96\zeta_3(5N+22)\right]\frac{g^4}{432} + O(g^5)\end{aligned}$$

d-dimensional fixed points defined by $\beta(g_c) = 0$ giving Wilson-Fisher fixed point at leading order as

$$g_c = \frac{6\epsilon}{[N+8]} + O(\epsilon^2)$$

where $d = 4 - 2\epsilon$

The values of the renormalization group functions at g_c are known as critical exponents

$$\eta \;=\; \gamma_\phi(g_c) \;\;,\;\; \omega \;=\; \beta'(g_c)$$

Exponents depend on the (dimensionless) parameters of the underlying field theory such as d and N; can be expanded in terms of ϵ or $1/N$

Can expand perturbative exponents in powers of $1/N$ and also compute exponents directly at the WF fixed point in the large N expansion using the method developed by Vasiliev et al

Large N expansion of critical exponents at Wilson-Fisher fixed point is only possible as the two loop term of $\beta(g)$ is *linear* in N

This method is *not* applicable to the large N_c expansion in $SU(N_c)$ non-abelian gauge theories

Exponents η and ν evaluated to three orders in $1/N$ [Vasiliev, Honkonen, Pismak]

ω has been determined at $O(1/N^2)$ [Broadhurst, Kreimer, Gracey] and is the anomalous dimension of the σ^2 operator in d-dimensions

At $g_c = g_c(\epsilon, N)$ can expand with respect to $1/N$ rather than ϵ

N is dimensionless and it plays the role of a perturbative coupling constant in $2 < d < 4$ dimensions

Formally in $d = D - 2\epsilon$ dimensions

$$g_c \;=\; a_{11}\frac{\epsilon}{N} \;+\; \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} a_{ij}\frac{\epsilon^j}{N^i}$$

if the two loop term of the β-function is linear in N

Then exponents can be expanded as

$$\eta \;=\; \sum_{i=1}^{\infty} \frac{\eta_i(d)}{N^i}$$

Large N methods of Vasiliev et al allow one to compute η_i to several orders in $1/N$ by exploiting scaling behaviour of fields at the WF fixed point and universality

Large N method

In d-dimensions at WF fixed point the underlying interaction $\frac{1}{2}\sigma\phi^i\phi^i$ is key

Both theories are in the *same* $d > 2$ universality class with propagators given by

$$\langle\phi^i(-k)\phi^j(k)\rangle \sim \frac{A\delta^{ij}}{(k^2)^{\mu-\alpha}} \ , \ \langle\sigma(-k)\sigma(k)\rangle \sim \frac{B}{(k^2)^{\mu-\beta}}$$

at the WF fixed point where $d = 2\mu$, A and B are amplitudes and

$$\alpha = \mu - 1 + \frac{1}{2}\eta \ , \ \beta = 2 - \eta - \chi$$

Or

$$2\alpha + \beta = 2\mu - \chi$$

Determine expressions for large N exponents by self-consistently solving the skeleton Schwinger-Dyson equations at criticality

η

As an example

$$\eta_1 = -\frac{4\Gamma(2\mu-2)}{\Gamma(2-\mu)\Gamma(\mu-1)\Gamma(\mu-2)\Gamma(\mu+1)}$$

$$\eta_2 = \left[\left[1-\frac{2\mu(\mu-1)}{(\mu-2)}\right][\Psi(2\mu-2)+\Psi(2-\mu)-\Psi(\mu-2)-\Psi(2)] + \frac{2\mu(\mu-1)}{(2-\mu)} + \frac{\mu(3-\mu)}{2(\mu-2)^2} + \frac{(\mu^2+\mu-1)}{2\mu(\mu-1)}\right]\eta_1^2$$

with $\Psi(z) = \frac{d}{dz}\ln\Gamma(z)$

Expanding η in powers of $d = 2 + 2\varepsilon$ and $d = 4 - 2\epsilon$ separately find *exact* agreement with ϕ anomalous dimension in $O(N)$ nonlinear σ model and $O(N)$ ϕ^4 theory to the high loop order in which they are available including recent high loop $O(N)$ ϕ^4 renormalization group functions

Other exponents have similar properties which indicates consistency with universality

η_3 determined from the conformal bootstrap construction in d-dimensions

Ultraviolet completeness

Exponents can be expanded around dimensions other than four and relate to theories in the same universality class

Ultraviolet completeness in six dimensions is a scalar cubic theory [Klebanov et al]

Key ingredients are renormalizability and common interaction with connectivity to lower dimensional theories

$$L = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3$$

A price to pay is an additional coupling constant or spectator interaction

Renormalization group functions available at three loops [Klebanov et al] based on [McKane et al] as well as at four loops [Gracey] and are in agreement with the large N exponents

σ mass anomalous dimension in agreement with ω at $O(1/N^2)$

Going up

Can extend to higher dimensions. For instance, in eight and ten dimensions the equivalent Lagrangians are

$$
\begin{aligned}
L^{(8)} &= \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}(\Box\sigma)^2 + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box\sigma + \frac{1}{24}g_3^2\sigma^4 \\
L^{(10)} &= \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2}(\Box\partial^\mu\sigma)(\Box\partial_\mu\sigma) + \frac{1}{2}g_1\sigma\phi^i\phi^i + \frac{1}{6}g_2\sigma^2\Box^2\sigma \\
&\quad + \frac{1}{2}g_3\sigma(\Box\sigma)^2 + \frac{1}{24}g_4^2\sigma^3\Box\sigma + \frac{1}{120}g_5^3\sigma^5
\end{aligned}
$$

Penalties to pay are more couplings as well as a higher order pole propagator

Computations of low order perturbation theory indicate the WF fixed point is connected to the lower dimensional theories and agree with the large N exponents

There appears to be no limit to the tower of theories

There are structural similarities of eight dimensional scalar theory with six dimensional QCD

Perturbative techniques

Have carried out extensive perturbative checks for higher dimensional (massless) Lagrangians throughout

Requires the evaluation of n-point functions to various loop orders

Main technique was the integration by parts algorithm of Laporta where linear relations between scalar Feynman integrals contributing to a Green's function are constructed and then solved algebraically down to a basic set of master integrals

Master integrals can be determined by direct methods such as by Panzer's package HYPERINT

Alternatively exploit Tarasov's method of relating d-dimensional integrals to $(d+2)$-dimensional integrals; 2-point four dimensional massless masters are known to four loops [Baikov & Chetyrkin]

For latter method is based on representing master integral using Schwinger parameters α_i

For instance, a propagator of the form $1/(k_i^2)^{\nu_i}$ maps to a factor in the Schwinger parameter integrand of $\alpha_i^{\nu_i - 1}$

After momentum integration the graph polynomial $D(\alpha_i)$ appears in integrand as

$$\frac{1}{D(\alpha_i)^{d/2}} = \frac{D(\alpha_i)}{D(\alpha_i)^{(d+2)/2}}$$

Rearrangement produces an integrand factor which implies that the integral can correspond to a $(d+2)$-dimensional integral but which is different from the lower dimensional one

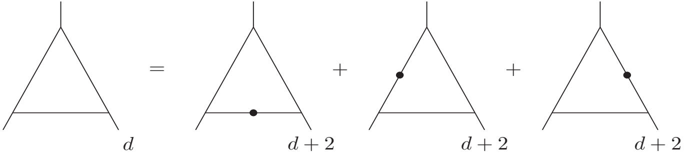
As $D(\alpha_i)$ is a polynomial in α_i then new numerator factor produces integrals with higher power propagators but with the same basic topology as the lower dimension graph

These integrals are reducible in the higher dimension by the Laporta method which means the higher dimensional (unknown) master is related to the known lower dimensional one plus simple already evaluated masters

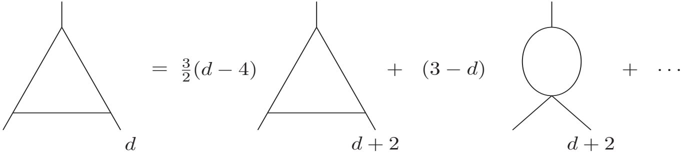
Example

Simple example is 3-point function at fully symmetric point

Applying Tarasov method gives



Applying Laporta reduction gives



Practical tools for designing automatic computations are:

- Feynman graphs generated using QGRAF
- Laporta integration by parts algorithm implemented in the C++ REDUZE encoding
- Internal algebra handled by symbolic manipulation language FORM

For the explicit perturbative computations for ultraviolet complete $O(N)$ scalar theories the 3-point and 4-point functions were evaluated at an off-shell point or a fully symmetric point; for higher n-point a vacuum bubble expansion was employed

Large N - new solutions

Re-examine the definition of the dimensions of the ϕ^i and σ fields from the $\frac{1}{2}\sigma\phi^i\phi^i$ vertex operator relation

$$2\alpha \; + \; \beta \; = \; 2\mu \; - \; \chi$$

More general solution is

$$\alpha \; = \; \mu \; - \; n \; + \; \tfrac{1}{2}\eta \;\; , \;\; \beta \; = \; 2n \; - \; \eta \; - \; \chi$$

for integer $n \geq 1$ which opens up a new set of theories which can be analysed by the large N expansion [JAG & Simms]

The integer n produces a higher derivative ϕ^i kinetic term

$$\bar{L}^{(4n)} \; = \; \frac{1}{2} \left(\partial_{\mu_1} \ldots \partial_{\mu_n} \phi^i \right)^2 \; + \; \frac{1}{2} \bar{\sigma} \phi^i \phi^i \; - \; \frac{1}{2g^2} \bar{\sigma}^2$$

which is renormalizable quartic $O(N)$ ϕ^4 theory in $4n$ dimensions

Interest in higher derivative theories due to connection with AdS/CFT and d-dimensional conformal field theories

Recent work for example by Brust and Hinterbichler on conformal algebraic structure of Green's functions of *free* higher derivative scalar theories

Now there is an opportunity to analyse interacting higher derivative $O(N)$ theories

Can construct new large N exponents based on $n = 2$ and 3 threads

$$
\begin{aligned}
\eta_1^{(8)} &= -\frac{6[\mu-1][\mu-4]\Gamma(2\mu-3)}{\Gamma^2(\mu+2)\Gamma(\mu)\Gamma(-1-\mu)} \\
\eta_2^{(8)} &= \left[-\frac{[2\mu^4 - 13\mu^3 - 2\mu^2 + 85\mu - 108]}{9[\mu-3][\mu-4]} [B(3-\mu) - B(\mu-1)] \right. \\
&\quad + \left[4\mu^{10} - 72\mu^9 + 433\mu^8 - 697\mu^7 - 3085\mu^6 + 15845\mu^5 \right. \\
&\quad \left. \left. \frac{- 26504\mu^4 + 11816\mu^3 + 15436\mu^2 - 16416\mu + 2592]}{18[\mu+1][\mu-1][\mu-2][\mu-3]^2[\mu-4]^2\mu} \right] {\eta_1^{(8)}}^2
\end{aligned}
$$

for $n = 2$

Natural question now is what are the fixed dimension field theories is the universality class of this thread

$n = 2$ thread Lagrangians

Algorithm to construct tower of theories is same for well-established $n = 1$ thread

Require common core force-matter interaction with ϕ^i kinetic term defining thread with independent σ spectator interactions included to ensure renormalizability in critical dimension; $[\sigma] = 4$ for $n = 2$

First few Lagrangians $L^{(d_1,d_2)}$ are

$$
\begin{aligned}
L^{(8,10)} &= \frac{1}{2}\left(\Box\phi^i\right)^2 + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{g_1}{2}\sigma\phi^i\phi^i \\
L^{(8,12)} &= \frac{1}{2}\left(\Box\phi^i\right)^2 + \frac{1}{2}(\Box\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^3 \\
L^{(8,14)} &= \frac{1}{2}\left(\Box\phi^i\right)^2 + \frac{1}{2}(\Box\partial^\mu\sigma)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^2\Box\sigma \\
L^{(8,16)} &= \frac{1}{2}\left(\Box\phi^i\right)^2 + \frac{1}{2}\left(\Box^2\sigma\right)^2 + \frac{g_1}{2}\sigma\phi^i\phi^i + \frac{g_2}{6}\sigma^2\Box^2\sigma \\
&\quad + \frac{g_3}{2}\sigma(\Box\sigma)^2 + \frac{g_4^2}{24}\sigma^4
\end{aligned}
$$

where d_1 is the critical dimension of the defining quartic theory and d_2 is the critical dimension of the Lagrangian in the tower

Results

Perturbative results established by application of earlier techniques such as Tarasov method and Laporta algorithm

Selection of β-functions to various orders

$$\begin{aligned}
\beta^{(8,10)}(g_1) &= [-N+6]\frac{g_1^3}{240} + [-197N-297]\frac{g_1^5}{288000} \\
&+ [-859789N^2 + 25272000\zeta_3 N - 38231814N \\
&\quad - 38232000\zeta_3 + 43101039]\frac{g_1^7}{43545600000} \\
\beta_1^{(8,12)}(g_1,g_2) &= [3Ng_1^2 + 40g_1^2 + 28g_1g_2 + 3g_2^2]\frac{g_1}{3360} \\
&+ [12042Ng_1^4 - 53464g_1^4 - 133308Ng_1^3g_2 \\
&\quad + 490392g_1^3g_2 - 14283Ng_1^2g_2^2 + 66956g_1^2g_2^2 \\
&\quad + 57960g_1g_2^3 - 42003g_2^4]\frac{g_1}{3556224000}
\end{aligned}$$

$$\begin{aligned}
\beta_1^{(8,14)}(g_1,g_2) &= [-18Ng_1^2 + 621g_1^2 - 252g_1g_2 + 7g_2^2]\frac{g_1}{272160} \\
&\quad + [171159480Ng_1^4 - 12056931g_1^4 - 67296096Ng_1^3g_2 \\
&\qquad - 377785296g_1^3g_2 + 1869336Ng_1^2g_2^2 - 7019838g_1^2g_2^2 \\
&\qquad - 59274432g_1g_2^3 + 1103396g_2^4]\frac{g_1}{186659085312000} \\
\beta_1^{(8,16)}(g_i) &= [45Ng_1^2 + 4356g_1^2 + 1584g_1g_2 + 792g_1g_3 + 296g_2^2 \\
&\quad - 384g_2g_3 + 159g_3^2]\frac{g_1}{11975040}
\end{aligned}$$

Determining the critical exponents from the renormalization group functions at the Wilson-Fisher fixed point, expanding in powers of ϵ within the large N expansion they are in agreement with the $n = 2$ thread large N exponents

Critical exponents derived from core $\sigma\phi^i\phi^i$ interaction present in large N universal theory contains information on the spectator interactions in fixed dimension Lagrangians

Similar analysis holds for $n = 3$ thread

Gross-Neveu model

Large N expansion with an underlying universal theory is not limited to scalar theories

Simple extension is to the $SU(N)$ Gross-Neveu model which is renormalizable asymptotically free two dimensional 4-fermi theory

$$L^{(2)}_{\text{GN}} \;=\; i\bar{\psi}^i \partial\!\!\!/ \psi^i \;+\; \frac{1}{2} g^2 (\bar{\psi}^i \psi^i)^2$$

or introducing an auxiliary field σ

$$L^{(2)}_{\text{GN}} \;=\; i\bar{\psi}^i \partial\!\!\!/ \psi^i \;+\; \frac{1}{2} g \sigma \bar{\psi}^i \psi^i \;-\; \frac{1}{2} \sigma^2$$

This formulation is parallel to the situation with four dimensional $O(N)$ ϕ^4 theory and is the basis for the large N fixed point construction of the d-dimensional critical exponents to three orders in $1/N$ [Vasil'ev et al; JAG]

Aside from the interest in d-dimensional conformal field theory two recent motivations for study of Gross-Neveu model are the connection with graphene and emergent supersymmetry in three dimensions [Giombi, Klebanov, Tarnopolsky]

At the Wilson-Fisher fixed point the $SU(4)$ theory lies in the chiral Ising universality class which is related to a particular electronic phase transition in the honeycomb lattice of graphene

Four loop renormalization of the $SU(N)$ Gross-Neveu model has recently been completed using vacuum bubble expansion [JAG, Luthe, Schröder]

The dimensionally regularized theory is not multiplicatively renormalizable; extra 4-fermi operators are generated such as

$$\mathcal{O}_n = \frac{1}{2} \bar{\psi}^i \Gamma_{(n)}^{\mu_1 \ldots \mu_n} \psi^i \, \bar{\psi}^i \Gamma_{(n) \; \mu_1 \ldots \mu_n} \psi^i$$

where

$$\Gamma_{(n)}^{\mu_1 \mu_2 \ldots \mu_n} = \gamma^{[\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_n]}$$

span the d-dimensional spinor space

These effect of these evanescent operators can be handled within the projection method of Bondi et al

Knowledge of the β-function

$$\beta(g) = (d-2)g - (N-1)\frac{g^2}{\pi} + (N-1)\frac{g^3}{2\pi^2} + (N-1)(2N-7)\frac{g^4}{16\pi^3} + (N-1)\left[-2N^2 - 19N + 24 - 6\zeta_3(11N-17)\right]\frac{g^5}{48\pi^4} + O(g^6)$$

means that $O(\epsilon^4)$ exponents can be computed; agrees with large N exponents

Summary of $SU(4)$ exponents for three dimensions

Exponent	2 loop	3 loop	4 loop	MC estimate
η_ψ	0.097	0.083	0.082	–
η_ϕ	0.906	0.778	0.745	0.754(8)
$1/\nu$	0.857	0.784	0.931	1.00(4)

MC from [Kärkkäinen et al]

Alternative is come down from the four dimensional Gross-Neveu-Yukawa theory

$$L^{(4)}_{\text{GN}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}g_1\sigma\bar{\psi}^i\psi^i + \frac{1}{24}g_2^2\sigma^4$$

Advantage is no evanescent problems

Non-abelian Thirring model

Higher dimensional ideas extend to non-abelian gauge theories such as QCD or theories of spin-1 adjoint fields

In the large N_f expansion QCD is equivalent to the two dimensional non-abelian Thirring model (NATM) at the Wilson-Fisher fixed point [Hasenfratz & Hasenfratz]

$$L^{\text{NATM}} = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \frac{\tilde{g}}{2}\left(\bar{\psi}^i T^a \gamma^\mu \psi^i\right)^2$$

Or using a spin-1 auxiliary field $A^a_\mu \propto \bar{\psi}^i T^a \gamma_\mu \psi^i$

$$L^{\text{NATM}} = i\bar{\psi}^i D\!\!\!\!/ \psi^i - \frac{1}{2} A^a_\mu A^{a\,\mu}$$

There are similar properties between these two models as there are between the scalar theories

Early large N_f QCD work extended that of Palanques-Mestre and Tarrach for QED

In d-dimensions $2 < d < 4$ the interaction is common and the NATM and QCD are in the same universality class at the Wilson-Fisher fixed point

For large N_f expansion use the NATM but with gauge fixing; the gluonic operators define the dimensionality of the coupling constants

Triple and higher gluon interactions emerge from integrating over closed quark loops [Hasenfratz & Hasenfratz]

Scaling propagators in Landau gauge are used for large N_f

$$\langle\psi(-p)\bar{\psi}(p)\rangle \sim \frac{A\not{p}}{(p^2)^{\mu-\alpha}} \;, \; \langle A_\nu(-p)A_\sigma(p)\rangle \sim \frac{B}{(p^2)^{\mu-\beta}}\left[\eta_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}\right]$$

Large N_f critical exponents in d-dimensions are in agreement with explicit perturbative renormalization group functions; useful checks on recent higher loop activity

Six dimensional QCD

Gauge invariant Lagrangian is constructed from all six dimensional operators which are independent; ignore operators which are total derivatives and not independent by Bianchi identity [D. Kazakov]

$$L^{(6)}_{\mathrm{GI}} = - \frac{1}{4} (D_\mu G^a_{\nu\sigma}) (D^\mu G^{a\,\nu\sigma}) + \frac{g_2}{6} f^{abc} G^a_{\mu\nu} G^{b\,\mu\sigma} G^{c\,\nu}{}_{\sigma} + i \bar{\psi}^{il} \not{D} \psi^{il}$$

Only two independent dimension six gluonic operators; could have used $\left(D^\mu G^a_{\mu\sigma}\right) (D_\nu G^{a\,\nu\sigma})$ alternatively to one of the above

Four-fermi operators are dimension 10 in six dimensions

Gauge fix in linear covariant gauge $\partial^\mu A^a_\mu = 0$ with gauge parameter α

Gauge fixing term has to be BRST invariant and dimension 6

$$L^{(6)}_{\mathrm{GF}} = - \frac{1}{2\alpha} (\partial_\mu \partial^\nu A^a_\nu) (\partial^\mu \partial^\sigma A^a_\sigma) - \bar{c}^a \Box (\partial^\mu D_\mu c)^a$$

Renormalization

Have renormalized massless six dimensional QCD to two loops in $\overline{\text{MS}}$

Similar to four dimensional renormalization but more integration due to higher pole propagators and extra quintic gluon self-interaction

$$
\begin{aligned}
\beta_1(g_1, g_2) &= [-249 C_A - 16 N_f T_F] \frac{g_1^3}{120} \\
&+ \Big[-50682 C_A^2 g_1^3 + 2439 C_A^2 g_1^2 g_2 + 3129 C_A^2 g_1 g_2^2 - 315 C_A^2 g_2^3 \\
&\quad - 1328 C_A N_f T_F g_1^3 - 624 C_A N_f T_F g_1^2 g_2 + 96 C_A N_f T_F g_1 g_2^2 \\
&\quad - 3040 C_F N_f T_F g_1^3\Big] \frac{g_1^2}{4320} \\
\beta_2(g_1, g_2) &= \Big[81 C_A g_1^3 - 552 C_A g_1^2 g_2 + 135 C_A g_1 g_2^2 - 15 C_A g_2^3 + 104 N_f T_F g_1^3 \\
&\quad - 48 N_f T_F g_1^2 g_2\Big] \frac{1}{120}
\end{aligned}
$$

Quark-gluon coupling constant is asymptotically free for all N_f; agreement with d-dimensional large N_f exponents

Fixed points

Can solve for fixed points in d-dimensions using $\beta_i(g_1, g_2) = 0$ and evaluate the exponents from the two loop renormalization group functions

For $N_f = 12$

$$\begin{aligned}
\gamma_A(g_1, g_2, 0)|_{(1)} &= 0.822064\epsilon - 0.071166\epsilon^2 + O(\epsilon^3) \\
\gamma_c(g_1, g_2, 0)|_{(1)} &= 0.088968\epsilon + 0.035583\epsilon^2 + O(\epsilon^3) \\
\gamma_\psi(g_1, g_2, 0)|_{(1)} &= -0.079083\epsilon + 0.129510\epsilon^2 + O(\epsilon^3) \\
\omega_+|_{(1)} &= 2.00000\epsilon - 0.479845\epsilon^2 + O(\epsilon^3) \\
\omega_-|_{(1)} &= 0.256816\epsilon + 0.215915\epsilon^2 + O(\epsilon^3) \\
\gamma_A(g_1, g_2, 0)|_{(3)} &= 0.822064\epsilon - 0.001544\epsilon^2 + O(\epsilon^3) \\
\gamma_c(g_1, g_2, 0)|_{(3)} &= 0.088968\epsilon + 0.000772\epsilon^2 + O(\epsilon^3) \\
\gamma_\psi(g_1, g_2, 0)|_{(3)} &= -0.079083\epsilon + 0.120155\epsilon^2 + O(\epsilon^3) \\
\omega_+|_{(3)} &= 2.000000\epsilon - 2.063346\epsilon^2 + O(\epsilon^3) \\
\omega_-|_{(3)} &= 0.283665\epsilon - 0.844111\epsilon^2 + O(\epsilon^3)
\end{aligned}$$

For $N_f \leq 16$ there are four non-trivial solutions

Two are stable and two are saddle points

One solution corresponds to $g_1 = 0$

Aside from this solution the $O(\epsilon)$ term of the critical exponents at each fixed point is the same since there is no $O(g_2)$ dependence at one loop

For $N_f > 16$ there are two real solutions and two complex conjugate ones

One of the real solutions is a saddle point and the other is stable

Stable solution in effect is that which corresponds to the large N_f critical exponents at the Wilson-Fisher fixed point

Eight dimensional QCD

Can extend the gauge theory tower to eight dimensions

$$
\begin{aligned}
L^{(8)} \quad = \quad & -\frac{1}{4}\left(D_\mu D_\nu G^a_{\sigma\rho}\right)\left(D^\mu D^\nu G^{a\,\sigma\rho}\right) + \frac{g_2}{4} f^{abc} G^a_{\mu\nu} D^\mu G^{b\,\sigma\rho} D^\nu G^c_{\sigma\rho} \\
& + i\bar{\psi}^{il} \not{D} \psi^{il} + \frac{g_3}{2} f^{abc} G^a_{\mu\nu} D^\sigma G^{b\,\mu\rho} D^\sigma G^{c\,\nu}_{\rho} + g_4^2 G^a_{\mu\sigma} G^{a\,\mu\rho} G^{b\,\sigma\nu} G^b_{\rho\nu} \\
& + g_5^2 G^a_{\mu\sigma} G^{b\,\mu\rho} G^{b\,\sigma\nu} G^a_{\rho\nu} + g_6^2 G^a_{\mu\sigma} G^a_{\nu\rho} G^{b\,\sigma\mu} G^{b\,\rho\nu} \\
& + g_7^2 G^a_{\mu\sigma} G^b_{\nu\rho} G^{a\,\sigma\mu} G^{b\,\rho\nu} + g_8^2 d_4^{abcd} G^a_{\mu\sigma} G^{b\,\mu\sigma} G^c_{\nu\rho} G^{d\,\nu\rho} \\
& + g_9^2 d_4^{abcd} G^a_{\mu\sigma} G^{c\,\mu\rho} G^{b\,\nu\sigma} G^d_{\nu\rho} + g_{10}^2 d_4^{acbd} G^a_{\mu\sigma} G^{b\,\mu\sigma} G^c_{\nu\rho} G^{d\,\nu\rho} \\
& + g_{11}^2 d_4^{adbc} G^a_{\mu\sigma} G^{c\,\mu\rho} G^{b\,\nu\sigma} G^d_{\nu\rho} \\
& - \frac{1}{2\alpha}\left(\partial_\mu \partial_\nu \partial^\sigma A^a_\sigma\right)\left(\partial^\mu \partial^\nu \partial^\rho A^a_\rho\right) - \left(\Box \bar{c}^a\right)\left(\Box \partial^\mu D_\mu c\right)^a
\end{aligned}
$$

where $d_4^{abcd} = d^{abe} d^{cde}$

One loop renormalization carried out separately for $SU(2)$, $SU(3)$ and $SU(N_c)$ due to properties of d^{abc}

Renormalization of 4-point operators was carried out by vacuum bubble expansion

Few β-functions are

$$
\begin{aligned}
\beta_1^{SU(2)}(g_i) &= \Big[24 N_f g_1^2 - 109 g_1^2 - 4158 g_1 g_2 - 1386 g_1 g_3 \\
&\quad + 567 g_2^2 + 378 g_2 g_3 + 63 g_3^2\Big] \frac{g_1}{3360} \\
\beta_2^{SU(2)}(g_i) &= \Big[-272 N_f g_1^3 + 32152 g_1^3 + 216 N_f g_1^2 g_2 + 17919 g_1^2 g_2 \\
&\quad - 19908 g_1^2 g_3 - 32634 g_1 g_2^2 - 2646 g_1 g_2 g_3 + 3528 g_1 g_3^2 \\
&\quad + 5103 g_2^3 + 2898 g_2^2 g_3 - 441 g_2 g_3^2 - 168 g_3^3\Big] \frac{1}{10080}
\end{aligned}
$$

Again renormalization group functions consistent with large N_f expansion

Eight dimensional QED

Can carry out higher loop analysis for the QED tower

$$
\begin{aligned}
L^{(8)} &= -\frac{1}{4}(\partial_\mu \partial_\nu F_{\sigma\rho})(\partial^\mu \partial^\nu F^{\sigma\rho}) - \frac{1}{2\alpha}(\partial_\mu \partial^\nu A_\nu)(\partial^\mu \partial^\sigma A_\sigma) \\
&+ i\bar{\psi}^i \not{D} \psi^i + \frac{g_2^2}{32} F_{\mu\nu} F^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} + \frac{g_3^2}{8} F_{\mu\nu} F^{\mu\sigma} F_{\nu\rho} F^{\sigma\rho}
\end{aligned}
$$

which has fewer quartic operators

$$
\begin{aligned}
\gamma_A(g_i,\alpha) &= \frac{N_f g_1^2}{35} + \frac{11 N_f g_1^4}{120} \ , \quad \beta_1(g_i) = \frac{N_f g_1^3}{70} + \frac{11 N_f g_1^5}{240} \\
\gamma_\psi(g_i,\alpha) &= [2\alpha + 7] \frac{g_1^2}{12} + [-964 N_f - 13475] \frac{g_1^4}{33600} \\
\beta_2(g_i) &= \left[1120 g_1^4 N_f + 72 g_1^2 g_2^2 N_f - 861 g_2^4 - 1659 g_2^2 g_3^2 - 609 g_3^4 \right] \frac{1}{1260} \\
\beta_3(g_i) &= \left[-1568 g_1^4 N_f + 144 g_1^2 g_3^2 N_f - 21 g_2^4 - 294 g_2^2 g_3^2 - 1029 g_3^4 \right] \frac{1}{2520}
\end{aligned}
$$

Renormalization group functions agree with large N_f QED critical exponents; spectator interactions in this and other Lagrangians play a key role in this comparison

Discussion

Have introduced new sets of scalar field theories which are connected across the dimensions via their respective Wilson-Fisher fixed point

These towers of theories have an underlying universal theory with an associated large N expansion; critical exponents and renormalization group functions are in accord

Ethos has been extended to other theories such as Gross-Neveu model and gauge theories

For spin-1 fields the tower begins with the non-abelian Thirring model and has been extended via QCD in four dimensions to six and eight dimensional gauge theories

At a deeper level would be to understand if there is a fundamental way of constructing the d- and $(d+2)$-dimensional equivalent theories without having to demonstrate it by explicit computations; is there a way of proceeding more fundamentally via a path integral approach?

REDUZE has tools to explore such structures