Effective actions for high energy scattering amplitudes in QCD and gravity

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1 Gluon reggeization

High energy amplitudes in QCD $(2E = \sqrt{s} \gg q = \sqrt{-t})$

$$M(s,t)|_{LLA}^{FKL} = M_{|_{Born}}\, s^{\omega(t)}\,,\;\; \alpha_s \ln s \sim 1\,,\;\; \alpha_s = \frac{g^2}{4\pi} \ll 1\,,$$

$$M_{|_{Born}} = 2\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{s}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}\,,\;\; [T^a, T^b] = i f_{abc} T^c$$

Effective gluon scattering vertex

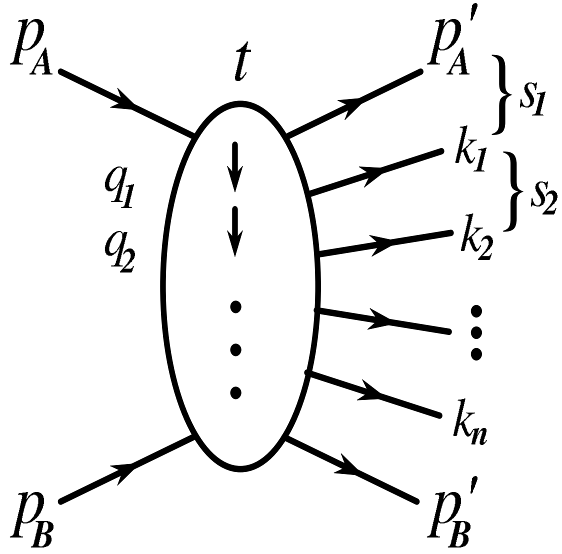
$$\gamma^B_{\mu'\mu} = -\delta_{\mu\mu'} + p_{\mu'}\frac{n^+_\mu}{p^+} + p'_\mu \frac{n^+_{\mu'}}{p^+} + q^2\, \frac{n^+_\mu n^+_{\mu'}}{2(p^+)^2} \to \delta_{\lambda'\lambda}\,,$$

$$n^+ = 2p_A/\sqrt{s}\,,\;\; n^- = 2p_B/\sqrt{s}\,,\;\; n^+ n^- = 2$$

Gluon Regge trajectory $\omega(t)$ in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k\, \frac{|q|^2}{|k|^2 |q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\, \ln \frac{|q^2|}{\lambda^2}$$

2 Gluon production at high energies



$$M^{FKL}_{2\to 2+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2 c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1} c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$C_\mu = -q^\perp_{1\mu} - q^\perp_{2\mu} + n^+_\mu \left(\frac{k_1^-}{2} + \frac{q_1^2}{k_1^+}\right) - n^-_\mu \left(\frac{k_1^+}{2} + \frac{q_2^2}{k_1^-}\right) \to C(q_2,q_1) = \frac{q_2^\perp\, q_1^{\perp *}}{k_1^{\perp *}}$$

3 BFKL Pomeron

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\;\sigma_t \sim s^{\Delta}\;,\;\;\Delta = \frac{4\alpha N_c}{\pi}\,\ln 2$$

Holomorphic separability (L. (1986))

$$H_{12} = h_{12} + h^*_{12}\,,\;\;[h_{12}, h^*_{12}] = 0\,,\;\;\rho_r = x_r + iy_r$$

Holomorphic BFKL Hamiltonian

$$h_{12} = \frac{1}{p_1}\ln(\rho_{12})\,p_1 + \frac{1}{p_2}\ln(\rho_{12})\,p_2 + \ln(p_1p_2) - 2\psi(1)\,,$$

Möbius invariance under $\rho_k \to \frac{a\rho_k+b}{c\rho_k+d}$ (L. (1986))

$$\Psi_{m,\widetilde{m}}(\vec{\rho}_1,\vec{\rho}_2;\vec{\rho}_0) = z^m z^{*\widetilde{m}}\,,\;\;z = \rho_{12}/(\rho_{10}\rho_{20})\,,$$

$$m = i\nu + \frac{1+n}{2}\,,\;\;\widetilde{m} = i\nu + \frac{1-n}{2}\,,\;\;\gamma = \frac{1}{2} + i\nu$$

4 BKP equation in LLA

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)\;,\;\; H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h + h^*)\,,\;\; [h, h^*] = 0\,,\;\; h = \sum_{k=1}^{n} h_{k,k+1}$$

Monodromy matrix and integrability (L. (1993))

$$t(u) = \prod_{k=1}^{n} \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}\,,\;\; [h, t_{11}(u) + t_{22}(u)] = 0$$

Integrability for composite states in adjoint representation (2009)

5 Effective theory for reggeized gluons

Gauge invariant effective lagrangian (L. (1995))

$$L_{eff} = L_{QCD} + Tr(V_+ \partial_\mu^2 A^+ + V_- \partial_\mu^2 A^-) + 2Tr\, \partial_\sigma^\perp A_+ \partial_\sigma^\perp A_-$$

Locality of field interactions in the rapidity space

$$|y - y_c| < \eta \ll \ln s\,, \;\; y = \frac{1}{2} \ln \frac{k_0 + k_3}{k_0 - k_3}$$

Constraints for reggeized gluon fields $A^\pm$

$$\delta A^\pm(x) = 0\,, \;\; \partial_\pm A^\pm(x) = 0$$

Eikonal representation for effective currents

$$V_\pm(\hat{v}_\pm) = \frac{1}{g} \partial_\pm O(x^\pm)\,, \;\; O(x^\pm) = -\frac{1}{D_\pm} \overleftarrow{\partial_\pm} = \frac{1}{1 - \frac{ig}{\partial_\pm} \hat{v}_\pm}$$

6 Principal value prescription for $V_\pm$

Symmetric choice for $1/\partial_\pm$ to provide the hermicity of L_{eff}

$$\frac{1}{\partial_\pm} f(x^\pm) = \frac{1}{4}\int_{-\infty}^{\infty} dy^\pm \, \epsilon(x^\pm - y^\pm)\, f(y^\pm)$$

Principal value prescription for the Green function

$$\epsilon(x^\pm - y^\pm) = P\int_{-\infty}^{\infty} \frac{dk^\mp}{4\pi i}\, \frac{e^{ik^\mp(x^\pm - y^\pm)}}{k^\mp}$$

Representation of $O(x^\pm)$ in terms of P-exponents

$$O(x^\pm) = \frac{P}{2}\, \frac{e^{i\frac{g}{4}\int_{-\infty}^{x^\pm} d\widetilde{x}^\pm v_\pm}}{e^{i\frac{g}{4}\int_{x^\pm}^{\infty} d\widetilde{x}^\pm v_\pm}} \left(P e^{-i\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x}^\pm v_\pm} + \bar{P} e^{i\frac{g}{4}\int_{-\infty}^{\infty} d\widetilde{x}^\pm v_\pm} \right)$$

Quasi-unitarity of $O(x^\pm)$

$$\frac{\partial}{\partial x^\pm} O^+(x^\pm) O(x^\pm) = 0$$

7 Classical equations for effective QCD

Euler-Lagrange equation for high energy QCD

$$[D_\mu, G^{\mu\nu}] = j^\nu\,,\; j^\pm = O(x^\pm)(\partial^2_\sigma A^\pm)O^+(x^\pm)\,,\; j^\nu_\perp = 0$$

Classical equation for a quasi-elastic kinematics

$$[D_\mu, G^{\mu\nu}] = O(x^+)\,\partial^2_{\perp\sigma}\, A^+(x^-, x_\perp)\; O^+(x^+)\,\delta^\nu_+$$

Transformation to the light-cone gauge $v'_+ = 0$

$$v'_\mu = V^{-1}(v_+)(v_\mu + \frac{\partial_\mu}{g})V(v_+)\,,\; V(v_+) = P\frac{e^{i\frac{g}{4}\int_{-\infty}^{x^\pm} d\widetilde{x}^\pm v_\pm}}{e^{i\frac{g}{4}\int_{x^\pm}^{\infty} d\widetilde{x}^\pm v_\pm}}$$

Classical solution as a superposition of shock waves

$$\tilde{v}_\nu(x) = \delta^-_\nu \int \frac{d^2z}{4\pi}\, \ln(|x-z|^2)\,\partial^{\perp 2}_\sigma\, A^+(x^-, z^\perp) = \delta^-_\nu\, A^+(x^-, x_\perp)$$

8 BFKL Pomeron and reggeized graviton at $N = 4$ SUSY

Diffusion approximation for the BFKL intercept

$$j = 2 - \Delta - \Delta\,\nu^2\,,\;\; \gamma = \frac{j}{2} + i\nu$$

AdS/CFT relation with the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\;\; t = E^2/R^2\,,\;\; \alpha' = \frac{R^2}{2}\,\Delta\,,\;\; \lambda = g^2 N_c$$

Anomalous dimension singularity at large coupling

$$\gamma = 1 - \sqrt{1 + (j-2)/\Delta}\,,$$

$$\Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4\,\lambda^{-3/2} - 2(1 + 3\zeta_3)\lambda^{-2}$$

Can one construct Gribov's reggeon calculus in N=4 SUSY?

9 High energy action in gravity

Effective action for the reggeized gravitons (L. 2011)

$$S=-\frac{1}{2\kappa^2}\int d^4x\left(\sqrt{-g}\,R+L_{ind}\right)\,,\;\delta g_{\mu\nu}=D_\mu\chi_\nu+D_\nu\chi_\mu$$

Induced lagrangian

$$L_{ind}=j_{++}\,\partial_\sigma^2 A^{++}+j_{--}\,\partial_\sigma^2 A^{--}+\partial_\sigma A^{++}\partial_\sigma A^{--}$$

Constraints for reggeized graviton fields

$$\delta A^{\pm\pm}(x)=0\,,\;\partial_\pm A^{\pm\pm}=0$$

Hamilton-Jacobi equation for currents

$$j^{\mp}=2x^{\mp}-\omega^{\mp}\,,\;g^{\mu\nu}\partial_\mu\omega^{\mp}\partial_\nu\omega^{\mp}=0\,,\;\omega^{\mp}=2x'^{\mp}$$

10 Global light-cone time systems

Coordinate transformation to the global light-cone system

$$g'^{\pm\pm} = g^{\mu\nu}\partial_\mu x'^{\pm}\,\partial_\nu x'^{\pm} = 0\,,\ \ x'^{\pm} = 2\omega^{\pm}$$

Coordinate transformation to the global time system

$$g'^{00} = g^{\mu\nu}\partial_\mu x^0\,\partial_\nu x^0 = 1$$

Schwarzschild metric with the mathematical time $\widetilde{t}$

$$dx^2 = (1 - \frac{2M}{r})d\widetilde{t}^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Transformation to the global time (Painleve (1921))

$$dx^2 = (1 - \frac{2M}{r})dt^2 + 2\sqrt{\frac{2M}{r}}dtdr - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

11 Classical equation for effective action

Einstein-Hilbert equation for effective gravity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \theta_{\mu\nu}(x)$$

Stress tensor in an arbitrary coordinate system

$$\theta_{\mu\nu} = \partial_\mu x'^+ \partial_\nu x'^+ \partial'^2_\chi A^{--}(x') + \partial_\mu x''^- \partial_\nu x''^- \partial''^2_\chi A^{++}(x'')$$

Coordinate transformations of the metric tensor

$$g'^{\rho\sigma} = g^{\mu\nu} \partial_\mu x'^\rho \partial x'^\sigma \,,\ g''^{\rho\sigma} = g^{\mu\nu} \partial_\mu x''^\rho \partial x''^\sigma$$

Constraints from the stress tensor conservation

$$D^\mu \theta_{\mu\nu} = 0 = g'^+_\nu g'^{\sigma+} \partial_\sigma \partial'^2_\chi A^{--}(x') + g'^-_\nu g'^{\sigma-} \partial_\sigma \partial''^2_\chi A^{++}(x'')$$

Self-consistency conditions for the metric tensor g'

$$g'^{\sigma+} \sim \eta^{\sigma+} \,,\ g''^{\sigma-} \sim \eta^{\sigma-} \,,\ \partial_\pm A^{\pm\pm} = 0$$

12 Restoration of the effective action

First variation of the induced action δS_{ind}

$$\int \frac{d^4x}{2\kappa^2}\sqrt{-g}\,\delta g^{\mu\nu}\left(\partial_\mu x'^{+}\partial_\nu x'^{+}\partial'^{2}_{\chi}A^{--}(x') + \partial_\mu x''^{-}\partial_\nu x''^{-}\partial''^{2}_{\chi}A^{++}(x'')\right)$$

Relations between variations

$$(\delta g^{\mu\nu})\partial_\mu x'^{+}\partial_\nu x'^{+} = -2\,g^{\mu\nu}\partial_\mu x'^{+}\partial_\nu \delta x'^{+} = -2\,g'^{\rho+}\partial'_\rho \delta x'^{+}\,,$$

$$(\delta g^{\mu\nu})\partial_\mu x''^{-}\partial_\nu x''^{-} = -2\,g^{\mu\nu}\partial_\mu x''^{-}\partial_\nu \delta x''^{-} = -2\,g''^{\rho-}\partial''_\rho \delta x''^{-}$$

Coordinate frame fixing

$$\sqrt{-g'}g'^{\rho+} = \eta^{\rho+}\,,\ \sqrt{-g''}g''^{\rho-} = \eta^{\rho-}$$

Restored induced action

$$S_{ind} = \int \frac{d^4x'}{2\kappa^2}\,\partial'_{-}\left(x'^{+} - \frac{\omega^{+}}{2}\right)\partial'^{2}_{\chi}A^{--} + \int \frac{d^4x''}{2\kappa^2}\,\partial''_{+}\left(x''^{-} - \frac{\omega^{-}}{2}\right)\partial''^{2}_{\chi}A^{++}$$

13 Solutions in quasi-elastic kinematics

Classical equation for $A^{++} = 0$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \partial_\mu x'^+ \partial_\nu x'^+ \partial'^2_\chi A^{--}(x')$$

A simple solution in the coordinate system x'

$$g_0'^{\rho\sigma} = \eta^{\rho\sigma} + \delta^\rho_- \delta^\sigma_- A^{--}(x')$$

Superposition of the solutions of Aichelburg and Sexl

$$g_0'^{\rho\sigma} = \eta^{\rho\sigma} - \delta^\rho_- \delta^\sigma_- \int \frac{d^2 z}{4\pi} \ln(|x - z|^2) \left(\frac{\partial}{\partial \vec{z}}\right)^2_\perp A^{--}(z^+, z^\perp)$$

Quantum fluctuations to calculate loop corrections

$$\delta g'^{\rho\sigma} = g'^{\rho\sigma} - g_0'^{\rho\sigma}$$

14 Multi-Regge processes in gravity

Multi-graviton production amplitude (L. (1982))

$$M_{2\to 2+n} \sim s^2 \kappa \delta_{\lambda_A \lambda_{A'}} \frac{s_1^{\omega_1}}{|q_1|^2} \kappa C(q_2, q_1) \ldots \kappa C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2} \kappa \delta_{\lambda_B \lambda_{B'}}$$

Graviton-graviton-reggeized graviton vertex

$$\gamma^{++}_{\mu'\nu',\mu\nu} = \gamma^{+}_{\mu'\mu} \gamma^{+}_{\nu'\nu} \to \delta_{\lambda\lambda'}$$

Reggeized graviton-reggeized graviton-graviton vertex

$$\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu - q_1^2 q_2^2 \left(\frac{n^+}{k^+} - \frac{n^-}{k^-}\right)_\mu \left(\frac{n^+}{k^+} - \frac{n^-}{k^-}\right)_\nu \to C(q_2, q_1)$$

Fulfilment of the Steinmann relation

$$\Delta_{s_1} \Delta_{s_2} M_{2\to 3} = 0$$

15 Graviton trajectory at supergravity

One loop graviton Regge trajectory (L. (1982))

$$j = 2 + \omega\,,\ \ \omega(q^2) = \frac{\alpha}{\pi}\int \frac{q^2\, d^2k}{k^2(q-k)^2} f(k,q)\,,\ \ \alpha = \frac{\kappa^2}{8\pi^2}\,,$$

$$f(k,q) = (k, q-k)^2 \left(\frac{1}{k^2} + \frac{1}{(q-k)^2}\right) - q^2 + \frac{N}{2}(k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \sum_{r=1}^{N} \bar{\psi}^r_\mu \gamma_5 \gamma_\nu \partial_\rho \psi^r_\sigma$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left(\ln\frac{|q|^2}{\lambda^2} + \frac{N-4}{2}\,\ln\frac{|\Lambda|^2}{|q|^2}\right)$$

16 Double-logarithms in gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha|q|^2 \ln \frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\;\; \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for super-gravity (BLS (2012))

$$f_\omega = 1 + \alpha|q|^2 \left(\frac{d}{d\omega}\,\frac{f_\omega}{\omega} - \frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\right)\,,\;\; \xi = \alpha\,|q|^2\,\ln^2\frac{s}{|q|^2}$$

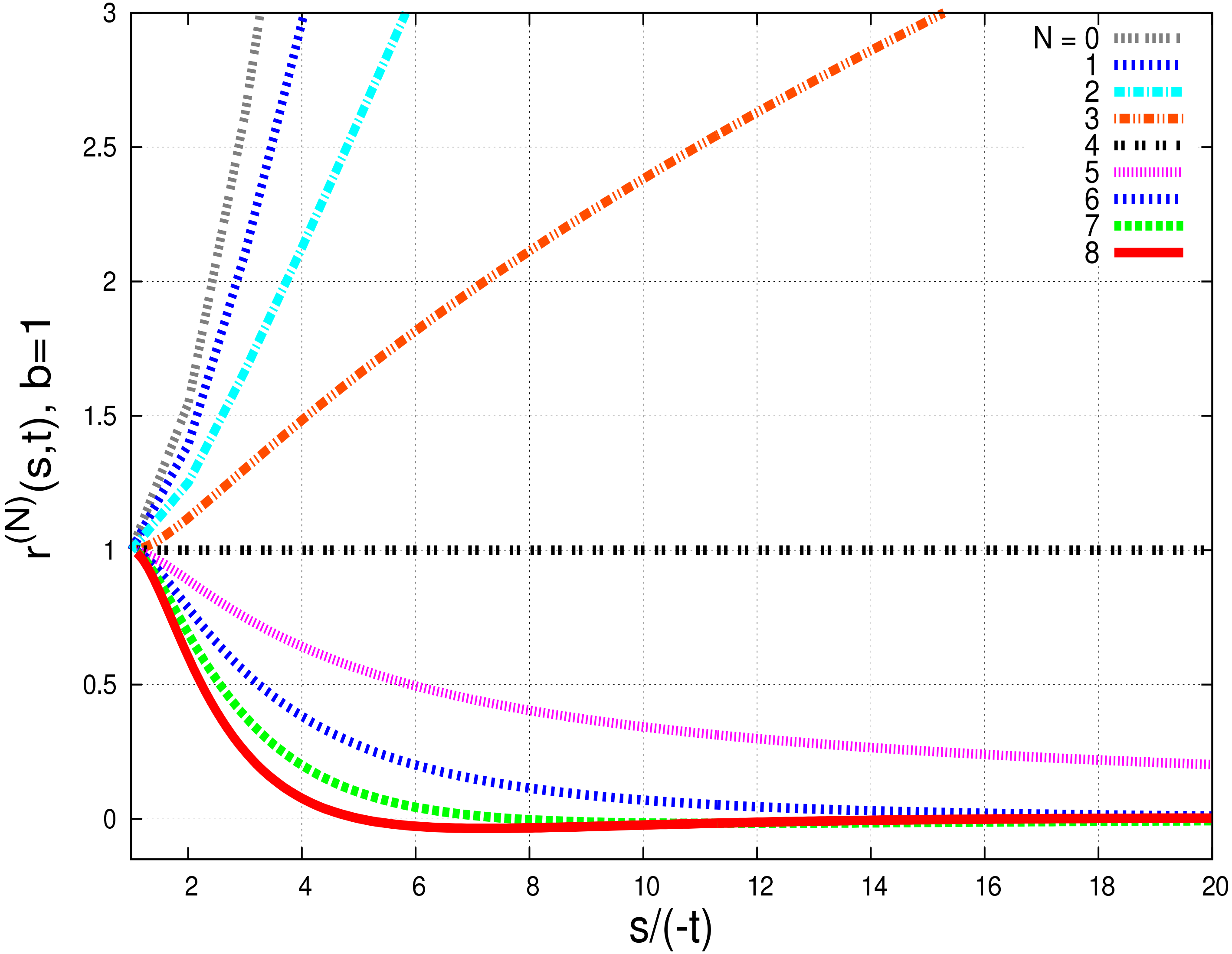
Its solution in terms of the parabolic cylinder function

$$\frac{f_\omega}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln\left(e^{\frac{x^2}{4}} D_{\frac{6-N}{2}}(x)\right)\,,\;\; x = \frac{\omega}{\sqrt{b}}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2-26N+36)\,\frac{\xi^3}{6!} + \ldots$$

17 Amplitudes in DL approximation



18 Discussion

1. Gluon and graviton are reggeons
2. BFKL dynamics and integrability
3. Effective action for high energy QCD
4. Euler-Lagrange equation for effective QCD
5. Pomeron and reggeized graviton in $N = 4$ SUSY
6. Effective action for reggeized gravitons
7. Euler-Lagrange equation for high energy gravity
8. Graviton Regge trajectory in supergravities
9. Scattering amplitudes in the DL approximation.