Université Pierre et Marie Curie - Paris 6

Laboratoire de Physique Théorique et Hautes Énergies (LPTHE)

Laboratoire de Physique Théorique Ecole Normale Supérieure (LPTENS)

"Séminaires Internationaux de Recherche de Sorbonne Universités"

Workshop on Multi-loop Calculations: Methods and Applications

Wednesday 7 and Thursday 8 of June 2017

Survey of multi-loop calculations in stochastic dynamics and fully developed turbulence

8 June 2017

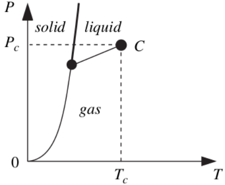
Tomáš Lučivjanský (Safarik University, Košice, Slovakia)

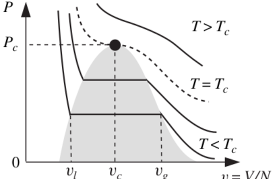
In cooperation with

- M. Hnatič
- L. Mižišin
- M. Dančo
- J. Honkonen
- N. V. Antonov
- N. M. Gulitskiy
- L. Ts. Adzhemyan
- M. Kompaniets

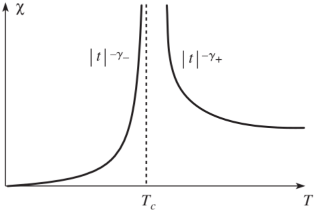
Motivation - static critical phenomena

phase transition liquid ⇔ vapour

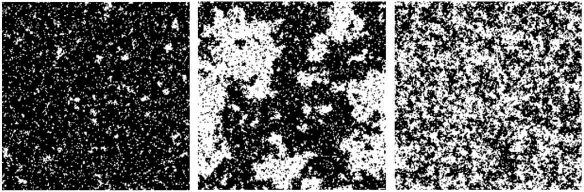




presence of divergences



2D Ising model
for $T = T_c$ fractal-like
behaviour



T=0.98 T_c T = T_c T = 1.05 T_c

Relevant concepts: order parameter, symmetry, spatial dimension, correlation length ξ diverges as $\tau^{-\nu}$, $\tau = (T - T_c)/T_c$ is a control parameter

Amit & Victor-Mayor 05

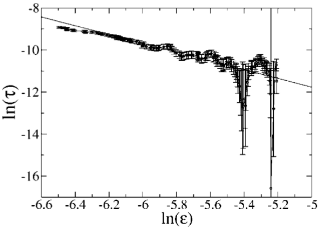
Motivation - dynamic critical phenomena

dynamic scaling

$$t_c(\tau) \sim (\xi(\tau))^z \sim |\tau|^{-z\nu}$$

scaling hypothesis

$$C(\tau, x, t) = |x|^{-(d-2+\eta)} \hat{C}\left(\frac{x}{\xi}, A\frac{t}{\xi^z}\right).$$



2D Ising ferromagnetic material
$z\nu = 2.09 \pm 0.06$
Dunlavy&Venus 2005

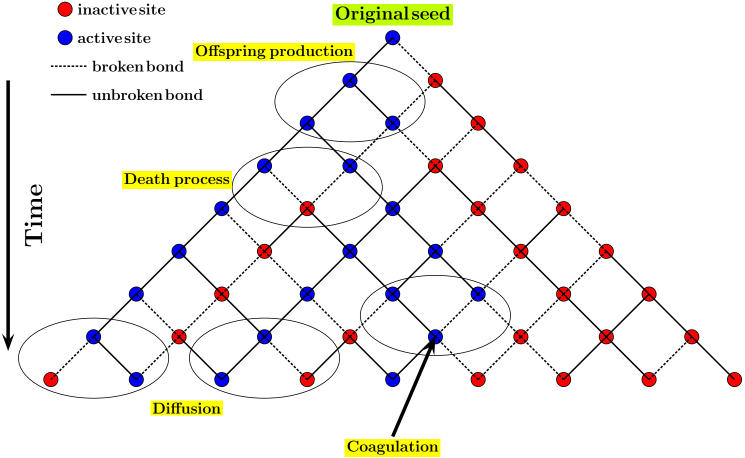
$$\partial_t \varphi_\alpha = \underbrace{F_\alpha[\varphi]}_{\text{reversible t.}} \underbrace{- D(i\boldsymbol{\nabla})^a \frac{\delta \mathcal{H}}{\delta \varphi_\alpha}}_{\text{dissipation}} + \underbrace{f_\alpha}_{\text{noise}}, \quad a \in \{0, 2\},$$

$$\langle f_\alpha \rangle = 0, \quad \langle f f \rangle = \underbrace{2k_B T D}_{\text{fluct.-dissip.}} (i\boldsymbol{\nabla})^a \delta(\ldots)$$

Vasil'ev 04, Folk & Moser 06, Täuber 14

Motivation - criticality in non-equilibrium systems

<u>Example:</u> directed percolation process (simple model for epidemics or in high energy physics - Gribov process)



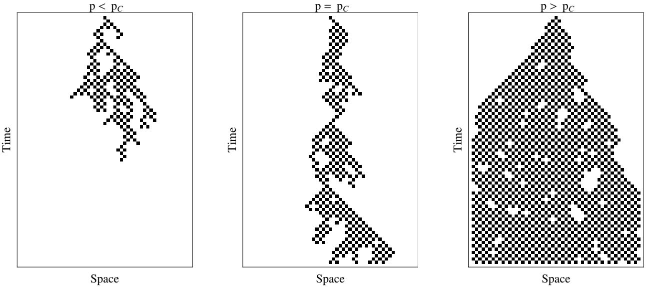
t	N(t)
1	1
2	1
3	2
4	2
5	2
6	3
7	4
8	3
9	3

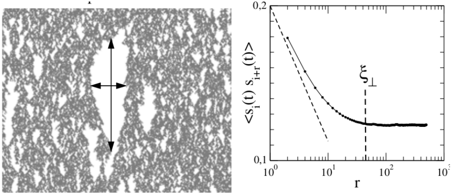
t - time in proper units $\qquad$ $N(t)$ - number of active sites

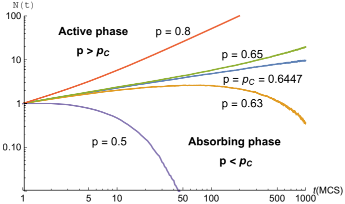
Not exactly solvable even in 1+1 dimension.

Motivation - criticality in non-equilibrium systems

detailed balance $\hat{H} \neq \hat{H}^\dagger$ is **violated**







correlation length $\xi \sim |p - p_c|^{-\nu_\perp}$

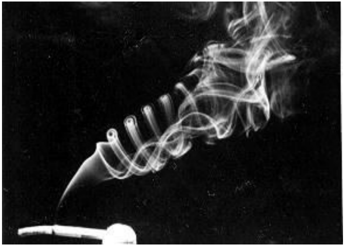
for $p = p_C$: $N(t) \sim t^\alpha$, otherwise $N(t) \sim \exp(\pm At)$

Velocity field - Nr. 1 (stochastic Navier-Stokes Eq.)

- stochastic equation $\partial_t \boldsymbol{v} + \boxed{(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}} = -\boldsymbol{\nabla} p + \boxed{\nu \nabla^2 \boldsymbol{v}} + \boldsymbol{f}$
- $\boldsymbol{f}$ - random force $\rightarrow$ input of energy

relevant physical parameter - **Reynolds** number

$$\mathrm{Re} = \frac{L_0 V_0}{\nu} = \frac{\boxed{\text{inertia}}}{\boxed{\text{dissipation}}}$$



- $\mathrm{Re} \ll 1$ - laminar flow
- fully developed homogeneous isotropic turbulence (in a statistical sense) for $\mathrm{Re} \rightarrow \infty$
- # of dof $\sim \mathrm{Re}^{9/4}$ Frisch 95

Vasil'ev 04

Velocity field - Nr. 1 (stochastic Navier-Stokes Eq.)

The equal-time structure functions

$$S_n(\boldsymbol{r}) = \left\langle \left[v_r(t, \boldsymbol{x}) - v_r(t, \boldsymbol{x}') \right]^n \right\rangle, \quad v_r = \boldsymbol{v} \cdot \boldsymbol{r}/r, \quad r \equiv |\boldsymbol{x} - \boldsymbol{x}'|$$

The Kolmogorov theory implies in the inertial range $l \ll r \ll L$

$$S_n(\boldsymbol{r}) = C_n \left(\mathcal{E} r \right)^{n/3}, \quad C_n \text{ universal}$$

$\mathcal{E}$ - the average power of the injection

in reality due to **intermittency** :

$$S_n(\boldsymbol{r}) = \left(\mathcal{E} r \right)^{n/3} \, (r/L)^{\gamma_n}$$

Theoretician's dream: to calculate γ_n within a controllable scheme.

Falkovich 01

Velocity field - Nr. 1 (stochastic Navier-Stokes Eq.)

Note on expansion scheme for turbulence problems

- Diagram technique of Wyld, Wyld 61
- stochastic force with the correlator

$$\langle f_i(\boldsymbol{k}) f_j(-\boldsymbol{k}) \rangle \propto k^{4-d-2y}$$

  expansion in formally small parameter y and spatial dimension d is fixed

- $y \to 2$ - energy doping from infinite scales

Adzhemyan et al. 98, Vasil'ev 04

Velocity field - Nr. 2 (Synthetic models)

Transport of some quantity - **observation** by R. Kraichnan (Kraichnan 68)

$$\partial_t \theta + \underbrace{(\boldsymbol{v} \cdot \boldsymbol{\nabla})\theta}_{\text{advection}} = \underbrace{D \boldsymbol{\nabla}^2 \theta}_{\text{diffusion}}$$

passive scalar might **be intermittent** for non-intermittent velocity field

Main idea: $\boldsymbol{v}(t, \boldsymbol{x})$ - random **Gaussian** variable $\langle \boldsymbol{v} \rangle = 0$ and

$$\langle \boldsymbol{v}(t, \boldsymbol{k}) \boldsymbol{v}(0, \boldsymbol{0}) \rangle \propto \theta(t) k^{2-d-y} P_{ij}^{k} \mathrm{e}^{-u_0 D_0 t k^{2-\eta}},$$

$P_{ij}^{k} = \delta_{ij} - k_i k_j / k^2$: incompressible part $(\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0)$

- **easy** generalizations - compressibility, anisotropy, helicity etc.
- **different** techniques: zero-mode approach, RG approach, $1/d$ expansion, numerical simulations

Antonov 06, Hnatic 16

Stochastic differential equations and quantum field theory

De Dominicis-Janssen dynamic functional

$$\partial_t \varphi(x) = U(x,\varphi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x'),$$
$$\Updownarrow$$
$$\mathcal{S}[\varphi] = \underbrace{\frac{1}{2}\varphi' D \varphi'}_{\text{noise properties}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

where the integrations over $t, \boldsymbol{x}$ have been omitted.

- Statistical averages as averages with a functional weight $\exp \mathcal{S}[\phi]$
- **quantum** field $\rightarrow$ **random classical** field
- Machinery of quantum **field theory applicable**: Feynman diagrammatic expansion, the renormalization group, operator product expansion etc.

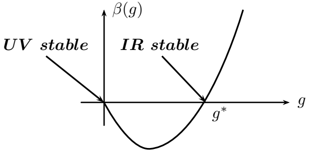
Vasil'ev 2004, Täuber 2014, Hnatic 16

Stochastic differential equations and quantum field theory

(1) proof of renormalizability

(2) Finding IR regimes

qualitative picture



$$\beta - \text{functions} \ \frac{\mathrm{d}\hat{g}}{\mathrm{d}\ln\mu} = -\beta(\hat{g})$$

Multi-charge theories $\beta_i(g_i^*) = 0$ and $\Omega_{ik} = \partial\beta_i/\partial g_k : |\Omega|_g^* > 0.$

(3) Experimentally measured quantities

$$F(x_1)F(x_2) = \sum_F C_F(\boldsymbol{r})F(t,\boldsymbol{x}), \quad \boldsymbol{x} = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2}, \quad \boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \to 0$$

in the inertial range $1/\mu \ll r \ll 1/m \sim L$

$$\langle F \rangle \propto m^{\Delta_F} \xrightarrow{\text{OPE}} \langle F_{nl}F_{ps} \rangle \approx \frac{(mr)^{\Delta_{n+p,0}}}{(\mu r)^{\Delta_{n,0}+\Delta_{p,0}}}$$

Vasil'ev 04, Amit 05

Differences

Main differences - turbulence problems

- **Majority** of calculations in the 2-loop approximation
- Adzhemyan et al., IJMP B 2003 - “One can say that **the two-loop** calculation for the stochastic NS equation is as **cumbersome** as the **four-loop** calculation for the conventional ϕ^4 model.”
- The **critical** dimensions (velocity and its powers, frequency, energy dissipation rate etc.) often given by the one-loop approximation **exactly**
- But, ε series for other important quantities **more complicated** - the calculation of the higher-order terms for them is of great interest, e.g. the correction exponent, the Kolmogorov constant, the inertial-range skewness factor

$$S_2(r) = C_k(\mathcal{E}r)^{2/3}, \qquad \mathcal{S} \equiv \frac{S_3}{S_2^{3/2}}$$

- higher-order terms **important** to judge about the validity and convergence of the perturbative expansion

Main differences

- To a given static model there are **various** dynamical extensions
  Hohenberg & Halperin 77

- **Multi-charge** theories

- Typical theory contains additional parameters. These are not charges, but the RG constants may depend on them.
  Example: model C (slow heat conduction) Vasil'ev 04

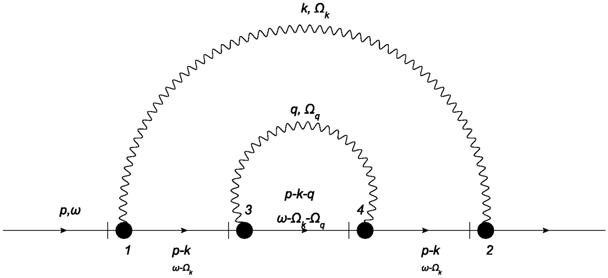
$$\mathcal{S} = \lambda_\psi \psi' \psi' + \psi'[-\partial_t + \lambda_\psi H_\psi] - \lambda_m m' \partial^2 m' + m'[-\partial_t - \lambda_m \partial^2 H_m],$$

$$H_\phi = \frac{\delta \mathcal{S}_{\text{stat}}}{\delta \phi},$$

$$\lambda_m = \lambda_\psi u \to Z_\lambda = 1 + \frac{g^2}{(1+u)\epsilon} + \dots$$

Main differences - an illustration for Feynman diagram

advection of a scalar quantity by the Navier-Stokes ensemble



$$\int \frac{\mathrm{d}\Omega_k}{2\pi}\frac{\mathrm{d}\Omega_q}{2\pi}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{[i(\omega-\Omega_k)+D(\boldsymbol{p}-\boldsymbol{k})^2]^2}\,\frac{1}{i(\omega-\Omega_k-\Omega_q)+D(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{q})^2}$$

$$\frac{d_f(q)P_{34}(\boldsymbol{q})}{\Omega_q^2+\nu_0^2 q^4}\,\frac{d_f(k)P_{12}(\boldsymbol{k})}{\Omega_k^2+\nu_0^2 k^4}(\boldsymbol{p}-\boldsymbol{k})_3(\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{q})_4\,\boldsymbol{p}_1(\boldsymbol{p}-\boldsymbol{k})_2 =$$

$$\frac{p^2}{4\nu_0^5(1+u)^2(2\pi)^{2d}}\int\frac{\mathrm{d}^d k}{k^6}\int\frac{\mathrm{d}^d q}{q^4}\,\boxed{\frac{q^2k^2-(\boldsymbol{q}\cdot\boldsymbol{k})^2}{k^2+q^2+u(\boldsymbol{k}+\boldsymbol{q})^2}}\,d_f(q)d_f(k)$$

Main differences - structure of propagators

effect of strong anisotropy

$$\mathcal{S}[\Phi] = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + \nu_0 \boldsymbol{\nabla}^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{\nabla})^2 \right] \theta - \boldsymbol{v} D_v^{-1} \boldsymbol{v}/2.$$

$$\langle \theta\theta' \rangle_0 = \langle \theta'\theta \rangle_0^* = \frac{1}{-\mathrm{i}\omega + \nu_0 k^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{k})^2},$$

$$\langle \theta\theta \rangle_0 = \frac{C(k)}{|-\mathrm{i}\omega + \nu_0 k^2 + \chi_0 \nu_0 (\boldsymbol{n}\cdot\boldsymbol{k})^2|^2}.$$

More involved issues in Adzhemyan et al., PRE 00

Main differences - expansion schemes

- Honkonen & Nalimov 96 - $2D$ dimensional turbulence, expansion in y and $\Delta : d = 2 + 2\Delta$

$$\mathcal{S} = \frac{1}{2}\boldsymbol{v}' D \boldsymbol{v}' + \boldsymbol{v}'[-\partial_t \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \nu_0 \nabla^2 \boldsymbol{v}]$$

- new divergence present in $v'v'$

- $$\langle f_m(t,\boldsymbol{k}) f_n(t',\boldsymbol{k}')\rangle = P_{mn}(\boldsymbol{k})\delta(t-t')\delta(\boldsymbol{k}+\boldsymbol{k}')d_f(k)$$

  kernel function $d_f(k)$ generalized to

$$d_f(k) = d_{f_1}(k) + d_{f_2}(k) = g_{10}\nu_0^3 k^{4-d-y} + g_{20}\nu_0^3 k^2$$

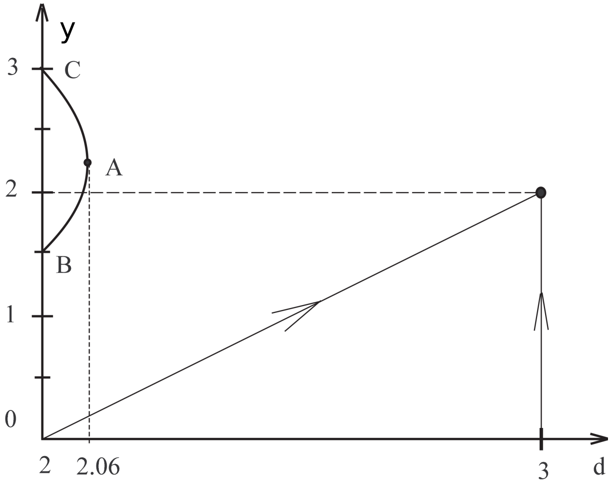
- double expansion in (Δ, y), where $\Delta = (d-2)/2$ is deviation from space dimension 2 and y - deviation from the Kolmogorov scaling

Hnatic et al. Acta Phys Slov 16

Main differences - expansion schemes

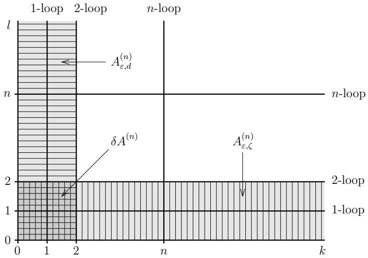
Ray scheme for 3D turbulence Adzhemyan et al. 05

$$y \propto \Delta = (d-2)/2$$



Main differences - expansion schemes

Adzhemyan et al. 10 - improved perturbation theory for 3D NS eq., details in Hnatič et al., Acta Phys Slov 16



$$A(y,d) = \sum_{k=0}^{\infty} A_k(d) y^k, \quad \zeta = y/\Delta$$

$$A^{(n)}_{eff} = A^{(n)}_{y,d} + A^{(n)}_{y,\zeta} - \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \zeta^{-k} a_{kl}\,\Delta^l\,,$$

	C_K	S
experiment	2.01	-0.28
1-loop	1.47	-0.45
double	1.889	-0.308

Overview

Overview - critical dynamics

Three-loop calculation for model A

CRITICAL DYNAMICS AS A FIELD THEORY

N. V. Antonov and A. N. Vasil'ev

Critical dynamics [1-3] is considered systematically from the point of view of quantum field theory. The connection between dynamics and statics and its consequences for the renormalization constants is discussed in detail. The main technical result is the calculation of the ε^3 contribution in the $4 - 2\varepsilon$ expansion of the dynamical exponent Δ_ω (critical dimension of frequency) for the O_n-symmetric φ^4 model. Instead of the value $\Delta_\omega = 2 + 0.726(1 - 2\varepsilon \cdot 1.687)\eta$ obtained previously [4], the value $\Delta_\omega = 2 + 0.726(1 - 2\varepsilon \cdot 0.1885)\eta$ is obtained.

- preprint Adzhemyan, Novikov & Sladkoff arXiv:0808.1347, Vestnik of St. Petersburg University **4** (4) (2008), 109-112. (reported 4-loop calculation)
- Adzhemyan, Borobeeva, Ivanova, Kompaniets "Theory withour divergences (in preparation)
- 5-loop calculations almost finished

Overview - turbulence problems

- **three-loop** calculation for Kraichnan model

PHYSICAL REVIEW E, VOLUME 64, 056306

Calculation of the anomalous exponents in the rapid-change model of passive scalar advection to order ε^3

L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits, and A. N. Vasil'ev
Department of Theoretical Physics, St. Petersburg University, Uljanovskaja 1, St. Petersburg-Petrodvorez 198504, Russia

(Received 11 June 2001; published 26 October 2001)

The field theoretic renormalization group and operator product expansion are applied to the model of a passive scalar advected by the Gaussian velocity field with zero mean and correlation function $\propto \delta(t-t')/k^{d+\varepsilon}$. Inertial-range anomalous exponents, identified with the critical dimensions of various scalar and tensor composite operators constructed of the scalar gradients, are calculated within the ε expansion to order ε^3 (three-loop approximation), including the exponents in anisotropic sectors. The main goal of the paper is to give the complete derivation of this third-order result, and to present and explain in detail the corresponding calculational techniques. The character and convergence properties of the ε expansion are discussed, the improved ''inverse'' ε expansion is proposed, and the comparison with the existing nonperturbative results is given.

- analytic results for anomalous exponents presented to ε^3
- It was shown that the knowledge of three terms allows one to obtain reasonable predictions for finite $\varepsilon \sim 1$; even the simple ε expansion captures some subtle qualitative features of the anomalous exponents established in numerical experiments

Overview - turbulence problems

- L.Ts. Adzhemyan, N.V. Antonov, J. Honkonen, PRE 02: ”Anomalous scaling of a passive scalar advected by the turbulent velocity field with finite correlation time: Two-loop approximation ”
- L.Ts. Adzhemyan, N.V. Antonov, M.V. Kompaniets, A.N. Vasil’ev, IJMP B 03: ”Renormalization-group approach to the stochastic Navier–Stokes equation: Two-loop approximation”
- Adzhemyan, Honkonen, Kompaniets, Vasil’ev, PRE 05 ”An improved ε expansion for three-dimensional turbulence: two-loop renormalization near two dimensions”
- L. Ts. Adzhemyan, N. V. Antonov, J. Honkonen, T. L. Kim, PRE 05: ”Anomalous scaling of a passive scalar advected by the Navier–Stokes velocity field: Two-loop approximation”
- work of M. Jurčišin and his group - variants of synthetic models with different generalizations

Overview - $1/d$ expansion turbulence

Main idea -

- multiscaling disappers for $d \to \infty$
- $1/d$ corrections calculable for passive advection
- combine y-expansion and large $d-$limit
- many integrals vanish
- expectation to obtiain simple result

$$u^* = \frac{8}{3}y - \frac{8}{9}y^2 - \frac{4}{9}y^3 + O(y^4),$$
$$u^* = \frac{8y}{3}(1 - 4y/3)^{1/4} (?)$$

- Adzhemyan, Antonov, Gol'din, Kim, Kompaniets, JPA 08 "Renormalization group in the infinite-dimensional turbulence: Third-order results"
- Adzhemyan, Antonov, Gol'din, Kompaniets, JPA 13: "Anomalous scaling of a passive vector field in d dimensions: Higher-order structure functions"

4-loop calculation under construction

Overview - Magnetohydrodynamics

- astrophysical applications Moffat 78, Shore 07
- **The magnetohydrodynamic limit**: the dense limit of a plasma motion of fluid given by the hydrodynamic equations and Ampères law connects the charge and bulk densities

$$\omega_p \tau_c \ll 1,$$

  ω_p the plasma oscillation time scale, τ_c the collision time
- two coupled stochastic equations

$$\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \kappa_0 \boldsymbol{\nabla}^2 \boldsymbol{B} + \boldsymbol{f}_\theta,$$
$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \nu_0 \boldsymbol{\nabla}^2 \boldsymbol{v} - \boldsymbol{\nabla} p + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{f}_v$$

- without feedback on velocity field $\rightarrow$ Kazantsev-Kraichnan model
- full 2-loop calculation still missing

Overview - compressible Navier-Stokes equation

The general stochastic NS eq.

$$\underbrace{\rho[\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}]}_{\text{convective term}} = \nu_0[\boldsymbol{\nabla}^2\boldsymbol{v} - \boxed{\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})}\,] + \boxed{\mu_0\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})} - \boldsymbol{\nabla}p + \boldsymbol{f},$$

ρ - the density, ν_0,μ_0 - molecular viscosities,

$\boldsymbol{v}$ - the velocity, $\boldsymbol{f}$ - random force.

$$\text{continuity equation} \qquad \partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0,$$

$$\text{thermodynamic considerations} \qquad p - p_0 = \boxed{c_0^2}(\rho - \rho_0).$$

$\boxed{c_0^2}$ - adiabatic speed of sound

- 1-loop $d = 3$ in Antonov 97
- around $d = 4$ within double expansion in Antonov 16
- $d = 2$ still lacking

Recent results

Directed bond percolation process

$$\mathcal{S}[\varphi] = \underbrace{\mathcal{S}_{\text{diff}}[\varphi]}_{\text{free part}} + \underbrace{\mathcal{S}_{\text{int}}[\varphi]}_{\text{interactions}}\ , \quad \varphi \equiv \{\psi, \tilde{\psi}\},$$

$$\mathcal{S}_{\text{diff}}[\varphi] = \int \mathrm{d}t \int \mathrm{d}^d x \Big\{ \tilde{\psi}[\partial_t - D_0 \boldsymbol{\nabla}^2 + D_0 \tau_0]\psi \Big\}, \qquad \overbrace{\boxed{\tau_0 \propto p - p_c}}^{\text{deviation from criticality}}\quad ,$$

$$\mathcal{S}_{\text{int}}[\varphi] = \int \mathrm{d}t \int \mathrm{d}^d x \Big\{ \frac{D_0 \lambda_0}{2}[\tilde{\psi}\psi^2 - \tilde{\psi}^2\psi] \Big\}.$$

expansion parameter $d = 4 - \varepsilon$

Honkonen: arXiv:1210.3934, Janssen& Täuber 2004

Directed bond percolation process

Critical exponents

a) The number $N(t,\tau)$ of active particles generated by a seed at the origin

$$N(t) = \int \mathrm{d}^d x \, G_{\psi\tilde{\psi}}(t, \boldsymbol{x}) \sim t^{-(\gamma_\psi + \gamma_{\tilde{\psi}})/\Delta_\omega}.$$

b) The mean square radius $R^2(t)$ of percolating particles, which started from the origin at time $t = 0$

$$R^2(t) = \frac{\int \mathrm{d}^d x \, \boldsymbol{x}^2 G_{\psi\tilde{\psi}}(t, \boldsymbol{x})}{2d \int \mathrm{d}^d x \, G_{\psi\tilde{\psi}}(t, \boldsymbol{x})} \sim t^{2/\Delta_\omega}.$$

c) Survival probability $P(t)$ of an active cluster originating from a seed at the origin

$$P(t) = -\lim_{k\to\infty} \langle \tilde{\psi}(-t, \mathbf{0}) \mathrm{e}^{-k \int \mathrm{d}^d x \; \psi(0, \boldsymbol{x})} \rangle \sim t^{-(d + \gamma_\psi + \gamma_{\tilde{\psi}})/2\Delta_\omega}.$$

Hinrichsen et al. 08

Directed bond percolation process

Sketch of the calculational procedure

- Normalization point
- Generation of topologies
- Construction of the Feynman diagrams
- Generation of the integrands
- Numerical calculations

based on Adzhemyan, Kompaniets, TMP 11 and Adzhemyan, Kompaniets, Novikov, Sazonov, TMP 13

# (loops)	# ($\Gamma_{\tilde{\psi}\psi}$)	# ($\Gamma_{\tilde{\psi}^2\psi}$)
1	1	1
2	2	11
3	17	150

Janssen&Täuber 05

- in preparation

Preliminary results

Critical exponents - the index η

$$\eta \equiv 2\gamma_\psi\big|_{g_2=g_2^*} = -\frac{\epsilon}{6}\boxed{-0.068077}\,\epsilon^2 + 0.0741655\epsilon^3 + O(\varepsilon^4).$$

Dynamic critical exponent z reads

$$z \equiv 2 - \gamma_D\big|_{u=u^*} = 2 - \frac{\varepsilon}{12}\boxed{-0.0292119}\,\varepsilon^2 + 0.0344042\varepsilon^3 + O(\varepsilon^3).$$

2-loop results Janssen 81, Bronzan 74

$$\eta = -\frac{\varepsilon}{6}\left[1 + \left(\frac{25}{288} + \frac{161}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right]$$
$$\approx -\frac{\varepsilon}{6}\boxed{-0.068075}\,\varepsilon^2 + O(\varepsilon^3),$$
$$z = 2 - \frac{\varepsilon}{12}\left[1 + \left(\frac{67}{288} + \frac{59}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right]$$
$$\approx 2 - \frac{\varepsilon}{12}\boxed{-0.0292091}\,\varepsilon^2 + O(\varepsilon^3).$$

Comparison of analytic results with numerical ones - difference smaller than 0.01%. Agreement with Janssen 2-loop results Janssen 81, Adzhemyan et al. 16

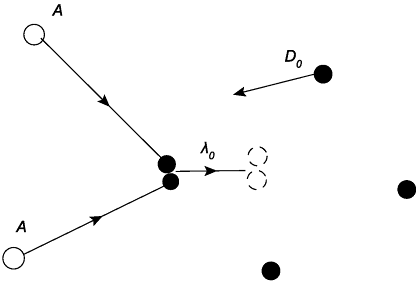
Exponent

	z		$\delta = \frac{d+\eta}{2z}$		$\theta = -\frac{\eta}{2z}$	
	$d=3$	$d=2$	$d=3$	$d=2$	$d=3$	$d=2$
T_2	1.8874	1.7164	0.7371	0.4486	0.1208	0.3167
T_3	1.9218	1.9917	0.7387	0.4989	0.0835	0.0061
P_1^1	1.8716	1.4424	0.7364	0.4428	0.1515	1.6709
P_2^1	1.9048	1.8091	0.7339	0.4187	0.0715	0.0651
P_1^2	1.9032	1.7985	0.7335	0.4077	0.1029	0.2197
[1] T_2	1.8874	1.7165	0.7371	0.4486	0.1208	0.3167
P_1^1	1.8716	1.4425	0.7364	0.4428	0.1515	1.6705
[2] Simul.	1.901(3)		0.732(4)		0.114(4)	
		1.765(2)		0.451(3)		0.229(3)

[1] Janssen 81, Bronzan 74 [2] Henkel et al. 08

Annihilation process $2A \to \varnothing$

particles are diffusing with diffusion constant D_0 and reacting after mutual contact



irreversible reaction $A + A \xrightarrow{\lambda_0} \varnothing$

particles A can be interpreted as molecules, biological entities, mutually annihilating random walk etc. continuum action S for $A + A \to \varnothing$ is

$$\mathcal{S} = \psi^\dagger[-\partial_t - (\boldsymbol{v}\cdot\boldsymbol{\nabla}) + D_0\nabla^2]\psi - \lambda_0 D_0[2\psi^\dagger + (\psi^\dagger)^2]\psi^2 - n_0\psi^\dagger|_{t=0} + \mathcal{S}_{\text{velocity}}$$

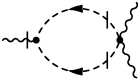
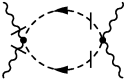
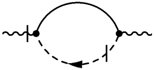
Annihilation process $2A \to \varnothing$

- Callan-Symanzik equation for the mean particle number

$$\left[(2-\gamma_1)t\frac{\partial}{\partial t} + \sum_g \beta_g \frac{\partial}{\partial g} - dn_0\frac{\partial}{\partial n_0} + d\right] n\left(t, \mu, \nu, n_0, g\right) = 0$$

- non-perturbative summation over n_0

- effective action

$\Gamma_R \;=\; \mathcal{S} + \frac{1}{4}$  $+\,\frac{1}{8}$  $+$  $+ \dots ,$

- stationarity equations

$$\frac{\delta\Gamma_R}{\delta\psi^\dagger} = \frac{\delta\Gamma_R}{\delta\psi} = 0$$

Hnatič et al. 2013

Annihilation process $2A \to \varnothing$

- density decay rate $n(t) \propto t^{-\alpha}$
- the decay exponent

$$\alpha = 1 + \frac{\gamma_4^*}{2 - \gamma_1^*}$$

where the anomalous dimensions γ_2 and γ_4 are defined as

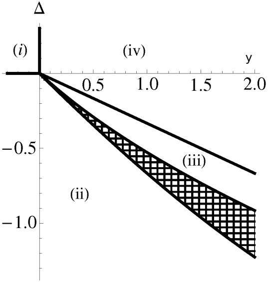
$$\gamma_2 = \frac{\partial \ln Z_2}{\partial \ln \mu}\bigg|_0, \quad \gamma_4 = \frac{\partial \ln Z_4}{\partial \ln \mu}\bigg|_0,$$

$$\gamma_2 = \gamma_D, \quad \gamma_4 = \gamma_\lambda - \gamma_D$$

Annihilation process $2A \to \varnothing$

Fixed point[1]	α	region of stability $\mathcal{O}(y,\Delta)$ [2]
Gaussian (i)	1	$y<0, \Delta>0$
Thermal (ii)	$1+\frac{\Delta}{2}+\frac{\Delta^2}{2}$	$\Delta<0, 2y+3\Delta<0$
Anomalous kinetic (iii)	$\frac{1+\Delta}{1-y/3}$	$y>0, -2y/3<\Delta<-y/3$
Normal kinetics (iv)	1	$y>0, \Delta>-y/3$
Driftless (v)	$1+\Delta$	unstable

Recall that $d = 2 + 2\Delta$



[1]for $d \le 2$ we have $n(t) \propto t^{-(1+\Delta)}$

[2]quadratic corrections are not presented

Annihilation process $2A \to \varnothing$

(i) Gaussian FP

- stable for $d > 2$ - mean field theory
- needed for the correct use of RG

(ii) Thermal FP

- local correlation stronger than long correlations and because $\Delta < 0$ ineq. $1 + \Delta/2 > 1 + \Delta$ holds
- at thermal point the decay is faster than $n \sim t^{-(1+\Delta)}$

(iii) Normal FP

- stable for $\Delta > -y/3$ with mean field-like behaviour $\alpha = 1$
- long range correlations destroy any effect of density fluctuations

(iv) Anomalous FP - here we have $1 + \Delta/2 < \alpha = (1 + \Delta)/(1 - y/3) < 1$

(v) unstable FP - realized when there is no stirring and thermal fluctuations

Model E of critical dynamics

Action functional

$$
\begin{aligned}
S[\Phi] = {} & 2\lambda_0 \psi^{\dagger\prime}\psi' - \lambda_0 u_0 m' \partial^2 m' \\
& + \psi^{\dagger\prime}\left[\left(-\partial_t + \lambda_0[\partial^2 - \tau_0 - \frac{g_{10}}{3}(\psi^\dagger\psi)]\right)\psi + i\lambda_0 g_{03}\psi\,(-m + h_0)\right] \\
& + \psi'\left[\left(-\partial_t + \lambda_0[\partial^2 - \tau_0 - \frac{g_{10}}{3}(\psi^\dagger\psi)]\right)\psi^\dagger - i\lambda_0 g_{03}\psi^\dagger\,(-m + h_0)\right] \\
& + m'\left[-\partial_t m + i\lambda_0 g_{03}(\psi^\dagger\partial^2\psi - \psi\partial^2\psi^\dagger) - \lambda_0 u_0 \partial^2(-m + h_{0m})\right].
\end{aligned}
$$

- universality class of planar magnet
- De Dominicis & Peliti 78 - a numerical error, later corrected Dohm 79
- Observation by Dohm confirmed by numerical means Adzhemyan et al. 16
- dimensional regularization with $d = 4 - 2\varepsilon$

Model E of critical dynamics

more convenient $f = \frac{g_3^2}{8\pi^2 u}$, $w = \frac{1}{u}$.
dynamical fixed point

$$f^* = (2\varepsilon) + \frac{3}{8}\left[2\ln\left(\frac{4}{3}\right) - 1\right](2\varepsilon)^2, \quad w^* = 1 + \frac{1}{50}\left[29 - 324\ln\left(\frac{4}{3}\right)\right](2\varepsilon).$$

$$\omega_f = (2\varepsilon) - 0.230097(2\varepsilon)^2, \quad \omega_w = \frac{(2\varepsilon)}{4} - 0.108171(2\varepsilon)^2$$

weak scaling fixed point

$$w^* = 0, \quad f^* = \frac{2}{3}(2\varepsilon) + \frac{2}{675}\left[702\ln\left(\frac{4}{3}\right) - 167\right](2\varepsilon)^2.$$

$$\omega_f = (2\varepsilon) - 0.126302(2\varepsilon)^2, \quad \omega_w = -\frac{(2\varepsilon)}{3} + 0.214675(2\varepsilon)^2$$

- Is is **unclear** which of two possible regimes is realized in real space dimension $d = 3$ - next order of perturbation theory is called for.
- at small ε the dynamical regime is IR-stable for $n = 2$.

List of potential future goals

- 3-loop stochastic Navier-Stokes equation
- 4-loop Kraichnan model/ 3-loop compressible version
- 2-loop compressible Navier-Stokes eq.
- 3-loop model E / 2-loop model E/F with activated velocity fluctuations
- 4-loop percolation problem
- 4-loop $1/d$ expansion in turbulence problems

References

- J. Cardy, *Scaling and Renormalization in Statistical Physics*, (Cambridge University Press, 2002).
- J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (4^{th} edition, Oxford University Press, Oxford, 2002).
- A. N. Vasil'ev *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics*, Boca Raton: Chapman & Hall/CRC (2004).
- U. Täuber, *Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling Behavior* (Cambridge University Press, New York, 2014).
- M. Hnatič, J. Honkonen, T. Lučivjanský, Acta Physica Slovaca **66**, 69 (2016).
- R. Folk, G. Moser, J. Phys. A: Math. Gen. **39**, R207 (2006).
- L. Ts. Adzhemyan and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998)
- P. Hohenberg and B. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
- H. K. Janssen, Z. Phys. B: Condens. Matter **23**, 377 (1976).
- C. De Dominicis, J. Phys. Colloq. France **37**, C1-247 (1976).
- H. K. Janssen, Z. Phys. B: Condens. Matter 42, 151 (1981)
- P. C. Martin, E. D. Siggia, H. A. Rose, Phys. Rev. A **8**, 423–437 (1973).
- U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

Thank you for your attention!