Five-loop quark mass and field anomalous dimensions for a general gauge group

Peter Marquard

DESY

PM, A.V. Smirnov, V.A. Smirnov, M. Steinhauser, D. Wellmann
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T. Luthe, A. Maier, PM, Y. Schröder
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Multi-Loop, Paris, June 2017

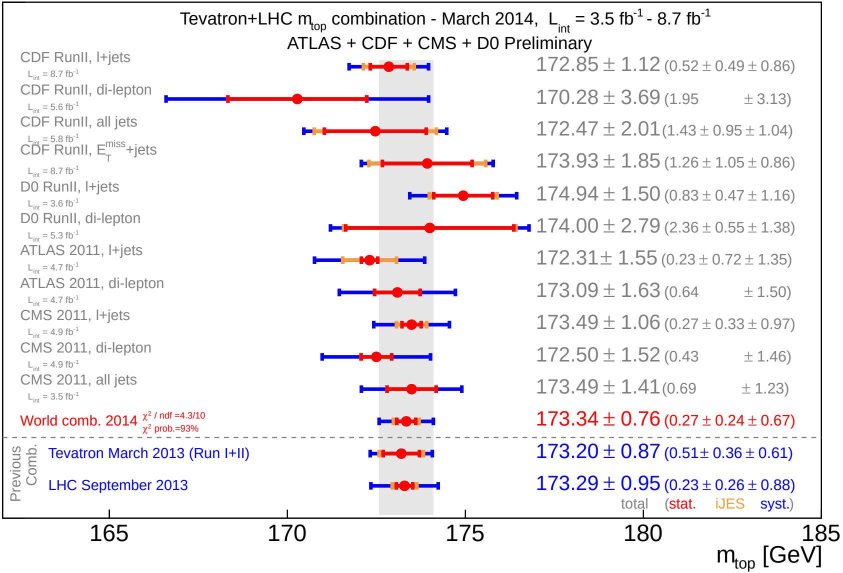
Outline

1. Mass relations
2. Quark mass and field anomalous dimensions
3. β function

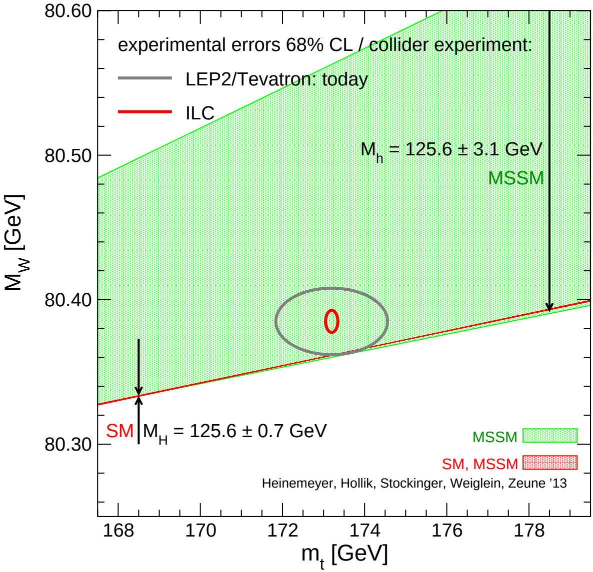
Introduction

- The renormalization constants are fundamental quantities of QCD.
- The β function governs the running of the coupling constant.
- The quark mass anomalous dimension together with the β function governs the running of the quark mass in the $\overline{\mathrm{MS}}$ scheme.
- Goal: Extend results in the literature to a general gauge group and perform an independent check.
- For heavy quarks several mass definitions exist, need translation rules.

LHC results for m_t



Precision tests



Outline

Scheme definitions

One considers the renormalized quark propagator

$$S_F(q) = \frac{-i\,Z_2}{\not{q} - Z_m m + \Sigma(q,m)}$$

with $\Sigma(q,m)$ the quark two-point function.
For the $\overline{\mathrm{MS}}$ scheme we require

$$S_F(q) \quad \text{finite}$$

and for the on-shell scheme we require a pole at the position of the mass

$$S_F(q) \overset{q^2 \to M^2}{\longrightarrow} \frac{-i}{\not{q} - M}$$

Even more quark mass definitions

- pole mass
- $\overline{\text{MS}}$ mass
- PS mass [Beneke '98]
- 1S mass [Hoang,Smith,Stelzer,Willenbrock '99]
- kinetic mass [Bigi,Shifman,Uraltsev,Vainstein '97]
- . . .

We need precise conversion formula for all these schemes!

Threshold mass schemes

- 1S mass

$$m^{1S} = M + \frac{1}{2} E_1^{pt}$$

$$E_1^{pt} = -\frac{C_F^2 M \alpha_s^2}{8} (1 + \alpha_s \ldots)$$

need binding energy @ three loops [Beneke et al]

Threshold mass schemes

- 1S mass

$$m^{\rm 1S} = M + \frac{1}{2} E_1^{\rm pt}$$

$$E_1^{\rm pt} = -\frac{C_F^2 M \alpha_s^2}{8}(1 + \alpha_s \ldots)$$

need binding energy @ three loops [Beneke et al]

- Potential-subtracted mass

$$m^{\rm PS} = M - \delta m(\mu_f)$$

$$\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}|<\mu_f} \frac{{\rm d}^3 q}{(2\pi)^3} V(\vec{q}) = \mu_f \frac{C_F \alpha_s}{\pi}(1 + \alpha_s \ldots)$$

need static potential @ three loops

[Smirnov,Smirnov,Steinhauser '09; Anzai,Kiyo,Sumino '09]

Setup of the calculation

- Need to calculate mass renormalization constant Z_m^{OS} by calculating four-loop on-shell integrals
- Together with the renormalization constant in the $\overline{\mathrm{MS}}$-scheme $Z_m^{\overline{\mathrm{MS}}}$

  [Chetyrkin '97; Larin, van Rittbergen, Vermaseren '97; Baikov,Chetyrkin,Kühn '14]

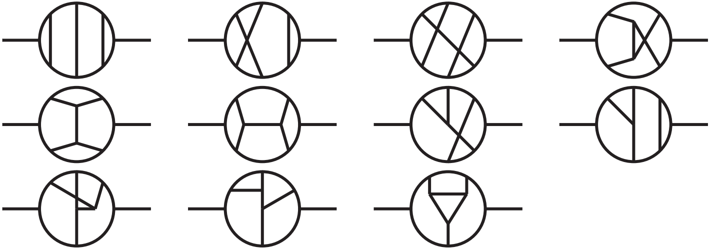
  we get

$$\left.\begin{array}{l} m_{\mathrm{bare}} = Z_m^{\mathrm{OS}} M \\ m_{\mathrm{bare}} = Z_m^{\overline{\mathrm{MS}}} m \end{array}\right\} \Rightarrow m = M \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}}$$

  - 1-loop [Tarrach'81]
  - 2-loop [Gray,Broadhurst,Grafe,Schilcher'90]
  - 3-loop [Chetyrkin,Steinhauser'99; Melnikov, v. Ritbergen'00; Marquard,Mihaila,Piclum,Steinhauser'07]

Setup of the calculation

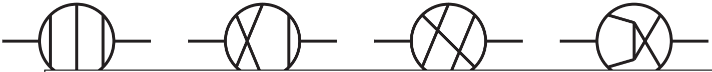
Need to calculate 4-loop on-shell diagrams of the form

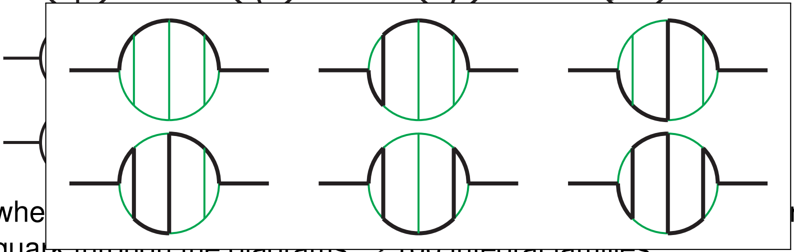


where we have to consider all possible ways to route the external quark through the diagrams $\Rightarrow$ 100 integral families.

Setup of the calculation

Need to calculate 4-loop on-shell diagrams of the form





whe … nal

quark through the diagrams ⇒ 100 integral families.

Setup of the calculation

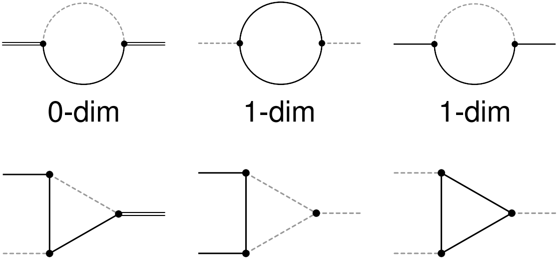
Follow the *standard* procedure for multi-loop calculations

- exploit that the appearing integrals are not linear independent but related through Integration-By-Parts identities [Chetyrkin, Tkachov '81]
- reduce all appearing integrals to a small set of basis integrals using `FIRE` [Smirnov] or `Crusher` [PM,Seidel]
- evaluate the remaining basis integrals ($\mathcal{O}(350)$) using analytic or numerical techniques (Mellin-Barnes or Sector Decomposition)[`MB.m` [Czakon] `FIESTA` [Smirnov]]

Mellin-Barnes

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}z\,\frac{Y^z}{X^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)\,,$$

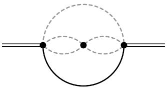
Calculate one-loop building blocks and insert into more complicated diagrams

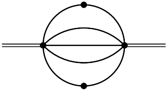


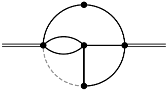
0-dim 1-dim 1-dim

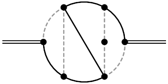
2-dim 3-dim 3-dim

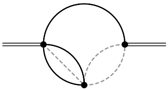
Mellin-Barnes Examples

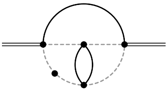

0-dim

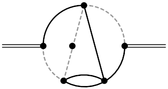

2-dim

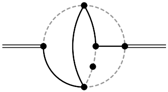

4-dim


6-dim


1-dim


3-dim


5-dim


7-dim

$\overline{\text{MS}}$–on-shell relation at four-loop order

$\overline{\text{MS}} \rightarrow$ on-shell

$$
\begin{aligned}
m_t(m_t) &= M_t \left(1 - 0.4244\,\alpha_s - 0.9246\,\alpha_s^2 - 2.593\,\alpha_s^3 \right. \\
&\quad \left. -(8.949 \pm 0.018)\,\alpha_s^4\right) \\
&= 173.34 - 7.924 - 1.859 - 0.562 \\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann ’16]

$\overline{\text{MS}}$–on-shell relation at four-loop order

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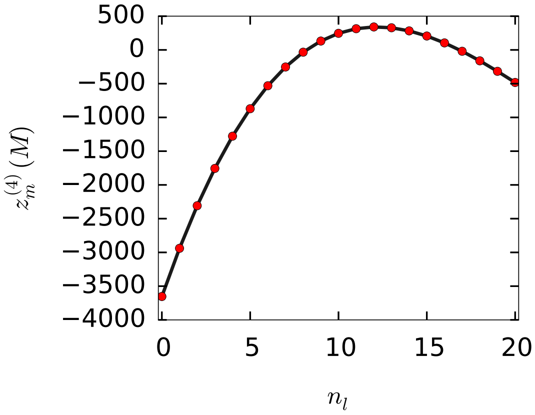
&= 173.34 - 7.924 - 1.859 - 0.562\\
&\quad - (0.209 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

[PM, Steinhauser, Smirnov, Smirnov, Wellmann '16]

$$
\begin{aligned}
M_b &= m_b(m_b) \left(1 + 0.4244\,\alpha_s + 0.9401\,\alpha_s^2 + 3.045\,\alpha_s^3 \right.\\
&\quad \left. +(12.685 \pm 0.025)\,\alpha_s^4\right)\\
&= 4.163 + 0.398 + 0.199 + 0.145 + (0.136 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

n_l dependence

$$
\begin{aligned}
z_m^{(4)} &= -3654.15 \pm 1.64 + (756.942 \pm 0.040)\, n_l \\
&\quad -43.4824\, n_l^2 + 0.678141\, n_l^3\,.
\end{aligned}
$$



N.B. Result available for arbitrary N_c and a general gauge group

PS mass $\leftrightarrow$ $\overline{\text{MS}}$ mass

$$
\begin{aligned}
m_t^{\text{PS}}(\mu_f = 80\ \text{GeV}) &= 163.508 + (7.531 - 3.685) \\
&\quad + (1.607 - 0.989) + (0.495 - 0.403) \\
&\quad + (0.195 - 0.211 \pm 0.0004)\ \text{GeV} \\
&= 163.508 + 3.847 + 0.618 + 0.092 \\
&\quad - (0.016 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_b^{\text{PS}}(\mu_f = 2\ \text{GeV}) &= 4.163 + 0.207 + 0.080 \\
&\quad + 0.032 - (0.0004 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

- large cancellations between contributions from OS-MS and PS-OS
- good convergence

1S mass $\leftrightarrow$ $\overline{\text{MS}}$ mass

$$
\begin{aligned}
m_t^{1S} &= 163.508 + (7.531 - 0.428) + (1.588 - 0.368) \\
&\quad + (0.479 - 0.262) + (0.185 - 0.174 \pm 0.0004)\ \text{GeV} \\
&= 163.508 + 7.103 + 1.220 \\
&\quad + 0.217 + (0.011 \pm 0.0004)\ \text{GeV}
\end{aligned}
$$

$$
\begin{aligned}
m_b^{1S} &= 4.163 + 0.352 + 0.123 \\
&\quad + 0.039 - (0.008 \pm 0.0003)\ \text{GeV}
\end{aligned}
$$

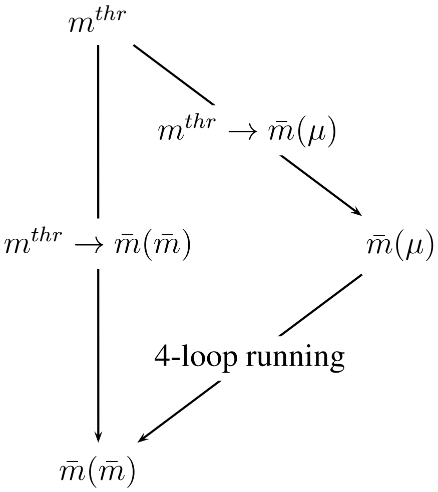
- large cancellations between contributions from OS-MS and 1S-OS
- good convergence

$m^{\text{thr}} \to \bar{m}(\bar{m})$

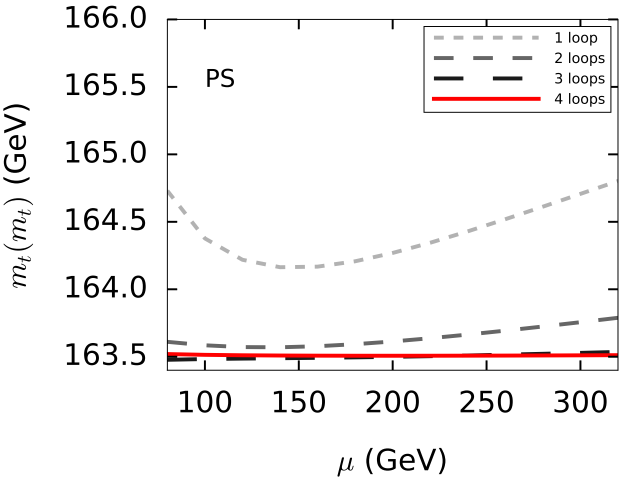
alternative way to obtain $\bar{m}(\bar{m})$

- first calculating $\bar{m}(\mu)$
- and in a second step $\bar{m}(\bar{m})$

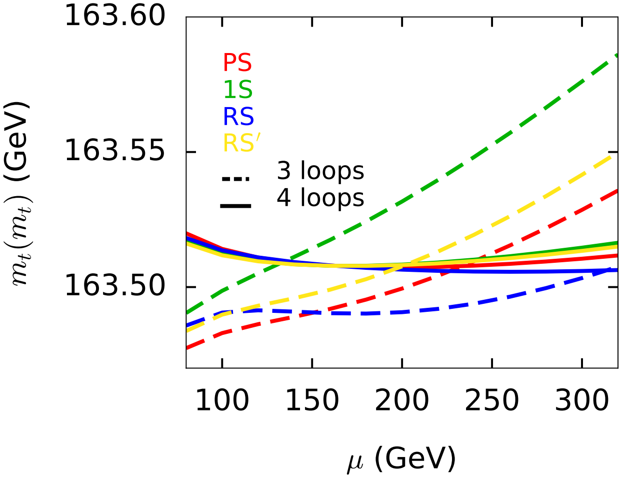
has to give the same result up to higher-order corrections



$m^{\rm thr} \rightarrow \bar{m}(\bar{m})$



$m^{\text{thr}} \to \bar{m}(\bar{m})$



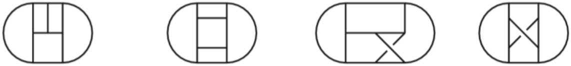
Outline

Method

- Need to calculate the relevant propagators and vertices at five-loop order.
- Renormalization constants (or anomalous dimensions) can be read off from the single pole.
- Only interested in the poles and since we work in the $\overline{\mathrm{MS}}$ scheme we can use a mass as infrared regulator and calculate fully massive five-loop tadpoles instead.
- Prize to pay: No multiplicative renormalization and a gluon mass counterterm.

Some details and statistics

- qq - 83,637, cc - 83,637, gg - 509,777, gcc - 1,444,756 five-loop diagrams generated with `QGRAF` [Nogueira '91]
- Mapped onto 4 12-line topologies



- Reduction to master integrals using IBPs in a Laporta-like implementation in `Crusher` [Marquard, Seidel] and `TIDE` [Luthe] with `Fermat` [Lewis] as backend
- Calculation done using `FORM` [Vermaseren]
- Color algebra done with `color` [van Ritbergen,Schellekens,Vermaseren '99]

Master integrals from Factorial Series

- The idea of the method goes back to Laporta who suggested to calculate Feynman integrals in form of of a factorial series. [Laporta '01]
- Take an integral and raise the power of one propagator to the power x e.g. $I(1,1,1) \to I(x) = I(x,1,1)$
- Using IBP relations one can obtain a difference equation for the integral

$$\sum_{k+0}^{R} p_k(x) I(x+k) = \sum_i \sum_{k=0}^{R_i} p_{ik}(x) J_i(x+k)$$

  where J_i are integrals of simpler sectors
- Make an ansatz for $I(x)$ in terms of a factorial series (N.B. not the most general one)

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$

Master integrals cont'd

- Inserting the ansatz into the difference equation results in a recurrence relation for a_s

$$\sum_{k=0}^{R'} g_k(s) a_{s+k} = \sum_i \sum_{k=0}^{R'_i} g_{ik}(s) a_{i,s+k}$$

- given the initial values $a_0, a_1, \ldots$ are known, an arbitrary number of values for a_n can be calculated.
- using the obtained values for a_n $I(x)$ can be calculated

$$I(x) = \sum_{s=0}^{\infty} \frac{\Gamma(x+1)}{\Gamma(x+d/2+s+1)} a_s$$
$$= \frac{\Gamma(x+1)}{\Gamma(x+d/2+1)} \Big(a_0 + \frac{a_1}{(x+d/2+1)} + \frac{a_2}{(x+d/2+1)(x+d/2+2)} + \cdots \Big)$$

Master integrals cont’d

- In this way we can obtain a numerical result for the master integrals with a very high precision
- For individual master integrals it is not always possible to reconstruct the analytic form since we do not know the alphabet
- For the final results for the anomalous dimensions the expressions are precise enough to reconstruct the analytic form in terms of ζ values using PSLQ

Quark anomalous dimensions

The anomalous dimensions are defined as usual

$$\gamma_2 = -\partial_{\ln\mu^2} \ln Z_2 =$$
$$-\, c_f\, a \left\{ (1-\xi) + \gamma_{21}\, a + \gamma_{22}\, a^2 + \gamma_{23}\, a^3 + \gamma_{24}\, a^4 + \dots \right\},$$
$$\gamma_m = \partial_{\ln\mu^2} \ln m_{\rm q}(\mu) \;=$$
$$-\, c_f\, a \left\{ 3 + \gamma_{m1}\, a + \gamma_{m2}\, a^2 + \gamma_{m3}\, a^3 + \gamma_{m4}\, a^4 + \dots \right\},$$
$$\text{with } a \equiv \frac{C_{\rm A}\, g^2(\mu)}{16\pi^2}$$

Known up to four loops from

[Tarrach '81; Tarasov; Larin '93; Larin, van Ritbergen, Vermaseren '97; Chetyrkin '97]

Colour factors

Besides the usual color factors

$$c_f = \frac{C_\mathrm{F}}{C_\mathrm{A}} \quad , \quad n_f = \frac{N_\mathrm{f}\, T_\mathrm{F}}{C_\mathrm{A}}$$

we also need

$$\begin{aligned} d_1 &= \frac{[\mathrm{sTr}(T^a T^b T^c T^d)]^2}{N_\mathrm{A}\, T_\mathrm{F}^2 C_\mathrm{A}^2} \, , \\ d_2 &= \frac{\mathrm{sTr}(T^a T^b T^c T^d)\, \mathrm{sTr}(F^a F^b F^c F^d)}{N_\mathrm{A}\, T_\mathrm{F} C_\mathrm{A}^3} \, , \\ d_3 &= \frac{[\mathrm{sTr}(F^a F^b F^c F^d)]^2}{N_\mathrm{A}\, C_\mathrm{A}^4} \end{aligned}$$

Color structures with a trace over 3 generators do not contribute.

γ_2: 2, 3 loop

The field anomalous dimension γ_2 is gauge dependent.

$$
\begin{aligned}
2^2\,\gamma_{21} &= n_f\Big[-8\Big] + \Big[-6c_f + 34 - 10\xi + \xi^2\Big]\,, \\
2^5 3^2\,\gamma_{22} &= n_f^2\Big[640\Big] + n_f\Big[8(108c_f - 1301 + 153\xi)\Big] \\
&+ \Big[432c_f^2 - 72(143 - 48\zeta_3)c_f + 2(10559 - 1080\zeta_3) \\
&\quad - 9\xi(371 + 48\zeta_3) + 27\xi^2(23 + 4\zeta_3) - 90\xi^3\Big]\,,
\end{aligned}
$$

γ_2: 4 loop

$$
\begin{aligned}
2^4 3^5\, \gamma_{23} = {} & n_f^3 \Big[13440 \Big] + n_f^2 \Big[6912(19 - 18\zeta_3) c_f \\
& + 16(6835 + 9072\zeta_3) + 64\xi(269 - 324\zeta_3) \Big] + n_f \Big[5184(19 - 48\zeta_3) c_f^2 \\
& + \big(-108(2407 - 1584\zeta_3 - 1296\zeta_4 - 5760\zeta_5) \\
& + 324\xi(767 - 528\zeta_3 - 144\zeta_4) \big) c_f + 497664 d_1 \\
& - (1365691 + 154224\zeta_3 + 97200\zeta_4 + 311040\zeta_5) \\
& + \xi(48865 + 152928\zeta_3 + 29160\zeta_4) - 54\xi^2(109 + 84\zeta_3 - 18\zeta_4) \Big] \\
& + \Big[-486(1027 + 3200\zeta_3 - 5120\zeta_5) c_f^3 + 324(5131 + 10176\zeta_3 - 17280\zeta_5) c_f^2 \\
& + \big(-108(23777 + 7704\zeta_3 + 2376\zeta_4 - 28440\zeta_5) - 1944\xi(6 - 7\zeta_3 + 10\zeta_5) \big) c_f
\end{aligned}
$$

γ_2: 4 loop cont'd

$$
\begin{aligned}
&+486\Big(16(-33+95\zeta_3-85\zeta_5)-8\xi(1+48\zeta_3-70\zeta_5)\\
&-8\xi^2(7\zeta_3+5\zeta_5)+20\xi^3(2\zeta_3-\zeta_5)\\
&-\xi^4(7\zeta_3-5\zeta_5)\Big)d_2+(10059589/4-241218\zeta_3+168156\zeta_4-604260\zeta_5)\\
&-\xi(2127929/8+164106\zeta_3-21141\zeta_4-107730\zeta_5)\\
&+27\xi^2(13883+9108\zeta_3-1548\zeta_4-1920\zeta_5)/8\\
&-81\xi^3(263+65\zeta_3-9\zeta_4+20\zeta_5)/2+81\xi^4(57+\zeta_3+10\zeta_5)/4\Big]\;.
\end{aligned}
$$

We obtained the result including the full ξ dependence.

γ_2: 5 loop

At five loop we only have the result in the Feynman gauge, $\xi = 0$

$$24^3\,\gamma_{24} = \frac{83 - 144\zeta_3}{72}\left[16n_f\right]^4 + \gamma_{243}\left[16n_f\right]^3 + \gamma_{242}\left[16n_f\right]^2 + \gamma_{241}\left[16n_f\right] + \gamma_{240} + \mathcal{O}(\xi)$$

For the coefficients γ_{24i} we obtain

$$\gamma_{243} = \left\{c_f, 1\right\}.\left\{-659/18 + 312\zeta_3 - 216\zeta_4, -3443/48 - 255\zeta_3 + 252\zeta_4\right\},$$

$$\gamma_{242} = \left\{c_f^2, c_f, d_1, 1\right\}.\left\{-2(2497 - 1200\zeta_3 + 3456\zeta_4 - 8640\zeta_5),\right.$$
$$477433/12 - 45636\zeta_3 + 4608\zeta_3^2 + 11448\zeta_4 - 65088\zeta_5 + 28800\zeta_6,$$
$$-384(115 - 33\zeta_3 - 90\zeta_5), 3015955/72 + 69509\zeta_3 - 2304\zeta_3^2$$
$$\left.-12861\zeta_4 + 16662\zeta_5 - 14400\zeta_6 - 11907\zeta_7\right\}$$

γ_2: 5 loop cont'd

$$\gamma_{241} = \left\{ c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1 \right\} . \left\{ 24(29209 + 89984\zeta_3 + 12288\zeta_3^2 - 28800\zeta_4 - 187520\zeta_5 + 76800\zeta_6), -4(296177 + 517020\zeta_3 + 26784\zeta_3^2 - 469908\zeta_4 - 4104720\zeta_5 + 1069200\zeta_6 + 3011904\zeta_7), -2304(748 + 4536\zeta_3 - 1368\zeta_3^2 - 6780\zeta_5 + 3255\zeta_7), 8(115334 - 37764\zeta_3 - 123012\zeta_3^2 - 49923\zeta_4 - 1124556\zeta_5 + 133650\zeta_6 + 1519308\zeta_7), 192(16732 + 39912\zeta_3 - 10944\zeta_3^2 - 72960\zeta_5 + 36771\zeta_7), 96(6158 - 13952\zeta_3 - 372\zeta_3^2 + 2880\zeta_4 - 39475\zeta_5 - 3900\zeta_6 + 45696\zeta_7), -34919359/9 - 753797\zeta_3 + 548148\zeta_3^2 - 135063\zeta_4 + 1759474\zeta_5 + 265350\zeta_6 - 2647806\zeta_7 \right\} ,$$

γ_2: 5 loop cont'd

$$\gamma_{240} = \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\right\}.\left\{1728(4977 + 128000\zeta_3 + 19968\zeta_3^2 + 180800\zeta_5 - 381024\zeta_7),\right.$$
$$-96(835739 + 8494144\zeta_3 + 1182336\zeta_3^2 - 316800\zeta_4 + 3983360\zeta_5 + 844800\zeta_6 - 17852688\zeta_7),$$
$$192(825361 + 5472068\zeta_3 + 651816\zeta_3^2 - 335808\zeta_4 - 1140420\zeta_5 + 950400\zeta_6 - 8056377\zeta_7),$$
$$4608(10 + 53226\zeta_3 - 15264\zeta_3^2 + 2145\zeta_5 - 45885\zeta_7),\ -16(84040774/9$$
$$+33396648\zeta_3 + 2804616\zeta_3^2 - 838782\zeta_4 - 18160944\zeta_5 + 6252300\zeta_6 - 41015331\zeta_7),$$
$$-384(43066 + 628802\zeta_3 - 160998\zeta_3^2 + 36540\zeta_4 - 201125\zeta_5 - 53475\zeta_6 - 403263\zeta_7),$$
$$-72(20566 - 218812\zeta_3 - 79080\zeta_3^2 - 13212\zeta_4 + 760220\zeta_5 + 20100\zeta_6 - 660667\zeta_7),\ 804023630/9$$
$$\left.+101490400\zeta_3 + 3143352\zeta_3^2 + 7356024\zeta_4 - 86186276\zeta_5 + 18372900\zeta_6 - 115799439\zeta_7\right\}.$$

in full agreement with [Baikov, Chetyrkin, Kühn '14] for $N_c = 3$

γ_m, 2 – 4 loop

The quark mass anomalous dimension is gauge independent

$$3^1\,\gamma_{m1} = n_f\Big[-10\Big] + \Big[(9c_f+97)/2\Big]\,,$$

$$3^3\,\gamma_{m2} = n_f^2\Big[-140\Big] + n_f\Big[54(24\zeta_3-23)c_f - 4(139+324\zeta_3)\Big]$$
$$+\Big[(6966c_f^2-3483c_f+11413)/4\Big],$$

$$3^4\,\gamma_{m3} = n_f^3\Big[-8(83-144\zeta_3)\Big] + n_f^2\Big[48(19-270\zeta_3+162\zeta_4)c_f$$
$$+2(671+6480\zeta_3-3888\zeta_4)\Big] + n_f\Big[-216(35-207\zeta_3+180\zeta_5)c_f^2$$
$$-3(8819-9936\zeta_3+7128\zeta_4-2160\zeta_5)c_f$$
$$-(65459/2+72468\zeta_3-21384\zeta_4-32400\zeta_5)+2592(2-15\zeta_3)d_1\Big]$$
$$+\frac{9}{8}\Big[-9(1261+2688\zeta_3)c_f^3+6(15349+3792\zeta_3)c_f^2$$
$$-2(34045+5472\zeta_3-15840\zeta_5)c_f+(70055+11344\zeta_3-31680\zeta_5)$$
$$-1152(2-15\zeta_3)d_2\Big]$$

γ_m, 5 loop

At five loops we get

$$6^5\,\gamma_{m4} = \gamma_{m44}\left[4n_f\right]^4 + \gamma_{m43}\left[4n_f\right]^3 + \gamma_{m42}\left[4n_f\right]^2 + \gamma_{m41}\left[4n_f\right] + \gamma_{m40}\ ,$$

with the coefficients

$$\gamma_{m44} = -6(65 + 80\zeta_3 - 144\zeta_4)\ ,$$

$$\gamma_{m43} = 3(4483+4752\zeta_3-12960\zeta_4+6912\zeta_5)c_f$$
$$+\,(18667/2+32208\zeta_3+29376\zeta_4-55296\zeta_5)\ .$$

$$\gamma_{m42} = \left\{c_f^2, c_f, d_1, 1\right\}.\left\{9(45253 - 230496\zeta_3 + 48384\zeta_3^2 + 70416\zeta_4\right.$$
$$+144000\zeta_5 - 86400\zeta_6),$$
$$375373 + 323784\zeta_3 - 1130112\zeta_3^2 + 905904\zeta_4 - 672192\zeta_5 + 129600\zeta_6,$$
$$-864(431 - 1371\zeta_3 + 432\zeta_4 + 420\zeta_5),$$
$$\left.4(13709 + 394749\zeta_3 + 173664\zeta_3^2 - 379242\zeta_4 - 119232\zeta_5 + 162000\zeta_6)\right\}\ ,$$

γ_m, 5 loop cont'd

$$\gamma_{m41} = \left\{c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1\right\}.\Big\{-54(48797 - 247968\zeta_3 + 24192\zeta_4 + 444000\zeta_5 - 241920\zeta_7),$$
$$-18(406861 + 216156\zeta_3 - 190080\zeta_3^2 + 254880\zeta_4 - 606960\zeta_5 - 475200\zeta_6 + 362880\zeta_7),$$
$$-62208(11 + 154\zeta_3 - 370\zeta_5),$$
$$753557 + 15593904\zeta_3 - 3535488\zeta_3^2 - 6271344\zeta_4 - 17596224\zeta_5 + 1425600\zeta_6 + 1088640\zeta_7,$$
$$1728(3173 - 6270\zeta_3 + 1584\zeta_3^2 + 2970\zeta_4 - 13380\zeta_5),$$
$$1728(380 - 5595\zeta_3 - 1584\zeta_3^2 - 162\zeta_4 + 1320\zeta_5),$$
$$-2(4994047 + 11517108\zeta_3 - 57024\zeta_3^2 - 5931900\zeta_4 - 15037272\zeta_5 + 4989600\zeta_6 + 3810240\zeta_7)\Big\}\ ,$$
$$\gamma_{m40} = \left\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\right\}.\Big\{972(50995 + 6784\zeta_3 + 16640\zeta_5),$$
$$-54(2565029 + 1880640\zeta_3 - 266112\zeta_4 - 1420800\zeta_5),$$
$$108(2625197 + 1740528\zeta_3 - 125136\zeta_4 - 2379360\zeta_5 - 665280\zeta_7),$$
$$373248(141 + 80\zeta_3 - 530\zeta_5),$$
$$-8(25256617 + 16408008\zeta_3 + 627264\zeta_3^2 - 812592\zeta_4 - 40411440\zeta_5 + 3920400\zeta_6 - 5987520\zeta_7),$$
$$-6912(9598 + 453\zeta_3 + 4356\zeta_3^2 + 1485\zeta_4 - 26100\zeta_5 - 1386\zeta_7),$$
$$5184(537 + 2494\zeta_3 + 5808\zeta_3^2 + 396\zeta_4 - 7820\zeta_5 - 1848\zeta_7),$$
$$4(22663417 + 10054464\zeta_3 + 1254528\zeta_3^2 - 1695276\zeta_4 - 41734440\zeta_5 + 7840800\zeta_6 + 5987520\zeta_7)\Big\}\ .$$

in full agreement with [Baikov, Chetyrkin, Kühn '14, Baikov, Chetyrkin, Kühn '17]

Outline

β function

We introduce the renormalization constants as

$$
\begin{aligned}
\psi_b &= \sqrt{Z_2}\psi_r \ , \quad A_b = \sqrt{Z_3}A_r \ , \quad c_b = \sqrt{Z_3^c}c_r \ , \\
m_b &= Z_m m_r \ , \quad g_b = \mu^\varepsilon Z_g g_r \ , \quad \xi_{L,b} = Z_\xi \xi_{L,r} \ ,
\end{aligned}
$$

or alternatively for the vertices

$$Z_1^j \text{ where } j \in \{3\text{g},4\text{g},\text{ccg},\psi\psi\text{g}\}$$

The anomalous dimensions are related through Ward identities

$$
\begin{aligned}
\gamma_3 &= 2(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \quad \gamma_1^{3g} &= 3(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \\
\gamma_1^{4g} &= 4(\gamma_1^{ccg} - \gamma_3^c) - \beta \ , \quad \gamma_1^{\psi\psi g} &= \gamma_1^{ccg} - \gamma_3^c + \gamma_2 \ ,
\end{aligned}
$$

and thus we choose to evaluate

$$Z_1^{ccg} = \sqrt{Z_3}\, Z_3^c\, Z_g$$

Ghost propagator: γ_3^c: full gauge dependence

$$
\begin{aligned}
\gamma_3^c &= -a\Big[-\tfrac{1}{4}(2+\xi)+\gamma_{31}^c a+\gamma_{32}^c a^2+\gamma_{33}^c a^3+\gamma_{34}^c a^4+\dots\Big] \\
{}^5\,3^1\,\gamma_{31}^c &= 5[16n_f]-2(98-3\xi)\;, \\
2^8\,3^3\,\gamma_{32}^c &= 35[16n_f]^2+\big(324(15-16\zeta_3)c_f+2(5+189\xi+1944\zeta_3)\big)[16n_f] \\
&\quad -4(14656+1485\xi-405\xi^2+81\xi^3)-648(4-\xi)(2-\xi)\zeta_3\;.
\end{aligned}
$$

γ_3^c: 4-loop, full gauge dependence

$$
\begin{aligned}
2^{11}3^4\gamma_{33}^c \quad = \quad & (83-144\zeta_3)[16n_f]^3 + \Big\{c_f, 1\Big\}.\Big\{24(1080\zeta_3-648\zeta_4-115), \\
& 2(779\xi-8315)/3 - 432(43+2\xi)\zeta_3 + 11664\zeta_4\Big\}[16n_f]^2 \\
& +\Big\{c_f^2, d_2, c_f, 1\Big\}.\Big\{-864(271+888\zeta_3-1440\zeta_5), 124416(4\zeta_3-5\zeta_5), \\
& 24\left(22517+3825\xi-864(43+\xi)\zeta_3+1296(23-\xi)\zeta_4-25920\zeta_5\right), \\
& 432(2983+42\xi-6\xi^2)\zeta_3 - 648(846-46\xi+\xi^2)\zeta_4 - 570240\zeta_5 \\
& +14(128354-722\xi-837\xi^2)/3\Big\}[16n_f] \\
& +\Big\{d_3, 1\Big\}.\Big\{1296(12(28-6\xi+\xi^2)-4(2392+108\xi-63\xi^2-17\xi^3+16\xi^4)\zeta_3 \\
& +5(1696+544\xi-252\xi^2+42\xi^3+7\xi^4)\zeta_5), \\
& -4(8202784+512546\xi-111402\xi^2+28107\xi^3-3888\xi^4)/3 \\
& -36(159040-19104\xi-162\xi^2+1092\xi^3-123\xi^4)\zeta_3 \\
& +1296(492-376\xi+91\xi^2-9\xi^3)\zeta_4 \\
& +270(28832+320\xi-732\xi^2+186\xi^3-7\xi^4)\zeta_5\Big\} .
\end{aligned}
$$

5-loop γ_3^c, Feynman gauge $\xi = 0$

$$
\begin{aligned}
2^{14}\,3^5\,\gamma_{34}^c &= \gamma_{344}^c\,[16n_f]^4 + \gamma_{343}^c\,[16n_f]^3 + \gamma_{342}^c\,[16n_f]^2 + \gamma_{341}^c\,[16n_f] + \gamma_{340}^c\,, \quad \gamma_{344}^c = 3(65 + 80\zeta_3 - 144\zeta_4)\,,\\
\gamma_{343}^c &= \Big\{c_f, 1\Big\}.\Big\{-2(14765 + 12528\zeta_3 - 38880\zeta_4 + 20736\zeta_5), -3(8325 + 15664\zeta_3 + 12240\zeta_4 - 33408\zeta_5)\Big\}\,,\\
\gamma_{342}^c &= \Big\{c_f^2, c_f, d_1, d_2, 1\Big\}.\Big\{-72(53927 - 182112\zeta_3 + 48384\zeta_3^2 + 42768\zeta_4 + 144000\zeta_5 - 86400\zeta_6),\\
&\quad -4(364361 + 484488\zeta_3 - 1804032\zeta_3^2 + 1868184\zeta_4 - 2239488\zeta_5 + 777600\zeta_6),\\
&\quad 20736(107 - 109\zeta_3 - 96\zeta_3^2 - 36\zeta_4 + 180\zeta_5), -41472(52\zeta_3 + 18\zeta_3^2 - 36\zeta_4 - 125\zeta_5 + 75\zeta_6),\\
&\quad 2(239495 - 3082212\zeta_3 - 1721088\zeta_3^2 + 3863376\zeta_4 - 156384\zeta_5 - 1425600\zeta_6)\Big\}\,,\\
\gamma_{341}^c &= \Big\{c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1\Big\}.\Big\{746496(7 + 26\zeta_3 + 490\zeta_5 - 560\zeta_7), 576(24617 - 301866\zeta_3\\
&\quad -196560\zeta_3^2 + 177066\zeta_4 + 274680\zeta_5 - 491400\zeta_6 + 725760\zeta_7), 165888(4 + 66\zeta_3 + 216\zeta_3^2\\
&\quad -705\zeta_5 + 357\zeta_7), 16(4796303 - 9571932\zeta_3 + 6399648\zeta_3^2 + 11100240\zeta_4 - 16127424\zeta_5 + 8845200\zeta_6\\
&\quad -10809288\zeta_7), -5184(4192 - 87152\zeta_3 + 21432\zeta_3^2 + 5616\zeta_4 + 89300\zeta_5 - 27300\zeta_6 - 20139\zeta_7),\\
&\quad -864(2805 - 86018\zeta_3 - 15960\zeta_3^2 + 43542\zeta_4 - 70360\zeta_5 - 68700\zeta_6 + 192906\zeta_7),\\
&\quad 2(52725013 + 136974540\zeta_3 + 1505088\zeta_3^2 - 118046052\zeta_4 - 226012536\zeta_5\\
&\quad +84380400\zeta_6 + 143718624\zeta_7)\Big\}\,,\\
\gamma_{340}^c &= \Big\{d_3, 1\Big\}.\Big\{-6912(5326 + 771746\zeta_3 - 17934\zeta_3^2 - 209916\zeta_4 - 1172870\zeta_5 + 377625\zeta_6\\
&\quad +396669\zeta_7), -8(192342607 + 174080040\zeta_3 + 36201384\zeta_3^2 - 103216464\zeta_4 - 855002232\zeta_5\\
&\quad +222650100\zeta_6 + 492202872\zeta_7)\Big\}\,,
\end{aligned}
$$

Ghost-gluon vertex: γ_1^{ccg}: full gauge dependence

$$
\begin{aligned}
\gamma_1^{ccg} &= -a(1-\xi)\left[\frac{1}{2} + \frac{6-\xi}{8}\,a + \gamma_{12}^{ccg} a^2 + \gamma_{13}^{ccg} a^3 + \gamma_{14}^{ccg} a^4 + \dots\right] , \\
2^7\,\gamma_{12}^{ccg} &= -15[16 n_f] + 2(250 - 59\xi + 10\xi^2) \, . \\
2^7\,3^5\,\gamma_{13}^{ccg} &= (-251 + 324\zeta_3)[16 n_f]^2 \\
&\quad + \big(324(96\zeta_3 + 36\zeta_4 - 161)c_f - 6166 + 4077\xi/2 \\
&\quad - 162(164 - 5\xi)\zeta_3 - 8748\zeta_4\big)[16 n_f] \\
&\quad + 1944\big((272 - 60\xi + 3\xi^2 + 7\xi^3)\zeta_3 \\
&\quad - 5(56 - 12\xi + 3\xi^2 + \xi^3)\zeta_5\big)\,d_3 \\
&\quad + 751120 - 27\xi(5434 - 1332\xi + 171\xi^2) \\
&\quad + 81(2528 - 548\xi + 99\xi^2 - \xi^3)\zeta_3 \\
&\quad + 1458(4-\xi)(2-\xi)\zeta_4 - 405(496 - 72\xi + 9\xi^2 + 2\xi^3)\zeta_5 \, .
\end{aligned}
$$

5-loop γ_1^{ccg}: Feynman gauge

$$
\begin{aligned}
2^{14}\,3^{5}\,\gamma_{14}^{ccg} &= \gamma_{143}^{ccg}\,[16n_f]^3 + \gamma_{142}^{ccg}\,[16n_f]^2 + \gamma_{141}^{ccg}\,[16n_f] + \gamma_{140}^{ccg}\,, \\
\gamma_{143}^{ccg} &= -2989 - 1440\zeta_3 + 5184\zeta_4\,, \\
\gamma_{142}^{ccg} &= \Big\{c_f, 1\Big\}.\Big\{1296(557 - 736\zeta_3 + 108\zeta_4 + 192\zeta_5), \\
&\quad 251891 + 1591056\zeta_3 - 335016\zeta_4 - 717984\zeta_5\Big\}\,, \\
\gamma_{141}^{ccg} &= \Big\{c_f^2, c_f, d_2, d_3, 1\Big\}.\Big\{5184(3731 + 9588\,zeta_3 - 1440\zeta_3^2 + 1332\zeta_4 - 10800\zeta_5 - 3600\zeta_6), \\
&\quad -1296(45129 - 14192\zeta_3 - 4032\zeta_3^2 + 5616\zeta_4 - 19296\zeta_5 - 7200\zeta_6), \\
&\quad -31104(1360\zeta_3 + 168\zeta_3^2 + 144\zeta_4 - 1260\zeta_5 - 300\zeta_6 - 441\zeta_7), \\
&\quad -10368(1126\zeta_3 + 150\zeta_3^2 - 567\zeta_4 - 1200\zeta_5 + 975\zeta_6 - 441\zeta_7), -42165410 \\
&\quad -432(145015\zeta_3 + 3564\zeta_3^2 - 9168\zeta_4 - 114001\zeta_5 - 10950\zeta_6 + 17640\zeta_7)\Big\}\,, \\
\gamma_{140}^{ccg} &= \Big\{d_3, 1\Big\}.\Big\{20736(70330\zeta_3 + 11076\zeta_3^2 - 8856\zeta_4 - 81380\zeta_5 + 16500\zeta_6 - 12607\zeta_7 - 2451), \\
&\quad 8(114251711 + 54643392\zeta_3 + 7060608\zeta_3^2 - 7531704\zeta_4 - 143288568\zeta_5 + 9023400\zeta_6 \\
&\quad +52599078\zeta_7)\Big\}\,.
\end{aligned}
$$

gluon propagator: γ_3: full gauge dependence

$$
\begin{aligned}
\gamma_3 &= a\Big[\tfrac{1}{6}(10+3\xi-8n_f)+\gamma_{31}\,a+\gamma_{32}\,a^2+\gamma_{33}\,a^3+\gamma_{34}\,a^4+\dots\Big]\\
2^3\gamma_{31} &= \Big\{c_f,1\Big\}.\Big\{-32n_f,-40n_f-2\xi^2+15\xi+46\Big\}\\
2^5 3^2\gamma_{32} &= \Big\{c_f^2,c_f,1\Big\}.\Big\{576n_f,1408n_f^2-16\,(432\zeta_3+5)\,n_f,\\
&\quad -54\,(\zeta_3+9)\,\xi^2+n_f\,(16\,(324\zeta_3-875)-576\xi)\\
&\quad +18\,(18\zeta_3+127)\,\xi+2432n_f^2-2\,(216\zeta_3-4051)+63\xi^3\Big\}
\end{aligned}
$$

γ_3: 4-loop, full gauge dependence

$$
\begin{aligned}
2^9 3^5 \gamma_{33} \quad = \quad & \Big\{ d_1\,, d_2, d_3, c_f^3\,, c_f^2\,, c_f\,, 1 \Big\} . \Big\{ 884736\,(24\zeta_3 - 11)\,n_f^2\,, -110592\,(516\zeta_3 + 135\zeta_5 - 64)\,n_f\,, \\
& -1944\,(8\zeta_3 + 5\zeta_5)\,\xi^4 + 3888\,(18\zeta_3 + 65\zeta_5)\,\xi^3 - 23328\,(21\zeta_3 + 55\zeta_5 - 1)\,\xi^2 \\
& +15552\,(278\zeta_3 - 9)\,\xi + 3456\,(1842\zeta_3 + 6030\zeta_5 - 131)\,, 5723136 n_f\,, -36864\,(264\zeta_3 - 169)\,n_f^2 \\
& -2304\,(5880\zeta_3 - 12960\zeta_5 + 10847)\,n_f\,, (5184\,(672\zeta_3 + 144\zeta_4 - 863)\,\xi \\
& -64\,(538272\zeta_3 - 244944\zeta_4 + 233280\zeta_5 - 363565)\,)\,n_f \\
& +1024\,(15768\zeta_3 - 5832\zeta_4 + 7541)\,n_f^2 + 630784 n_f^3\,, \\
& 81\,(74\zeta_3 - 115\zeta_5 - 360)\,\xi^4 + 162\,(36\zeta_3 - 36\zeta_4 + 325\zeta_5 + 1657)\,\xi^3 \\
& -324\,(1267\zeta_3 - 294\zeta_4 + 105\zeta_5 + 3823)\,\xi^2 + n_f\,(1296\,(32\zeta_3 \\
& -12\zeta_4 + 129)\,\xi^2 - 32\,(95904\zeta_3 + 12636\zeta_4 + 35345)\,\xi \\
& +32\,(1249020\zeta_3 - 376164\zeta_4 - 427680\zeta_5 - 1404961)\,) \\
& +n_f^2\,(256\,(1296\zeta_3 - 1229)\,\xi - 256\,(18360\zeta_3 - 17496\zeta_4 - 41273)\,) \\
& -2048\,(432\zeta_3 - 355)\,n_f^3 + 4\,(756216\zeta_3 - 141912\zeta_4 - 427680\zeta_5 + 1539403)\,\xi \\
& -32\,(338580\zeta_3 - 26973\zeta_4 - 415125\zeta_5 - 504770)\,\Big\}
\end{aligned}
$$

5-loop γ_3: Feynman gauge

$$
\begin{aligned}
2^9\,3^5\,\gamma_{34} &= \gamma_{344}\,[16n_f]^4 + \gamma_{343}\,[16n_f]^3 + \gamma_{342}\,[16n_f]^2 + \gamma_{341}\,[16n_f] + \gamma_{340} \\
\gamma_{344} &= \Big\{c_f\,,1\Big\}.\Big\{-2\,(144\zeta_3+107)\,,619-864\zeta_4\Big\} \\
\gamma_{343} &= \Big\{c_f^2\,,c_f\,,1,\,d_1\Big\}.\Big\{24\,(11424\zeta_3-4752\zeta_4-4961)\,,-4\,(63648\zeta_3-57024\zeta_4+20736\zeta_5+16973)\,, \\
&\quad -8\,(25992\zeta_3+5940\zeta_4-29376\zeta_5+14843)\,,-6912\,(123\zeta_3-36\zeta_4-60\zeta_5-55)\Big\} \\
\gamma_{342} &= \Big\{c_f^3\,,c_f^2\,,c_f\,,1,\,c_fd_1\,,d_1\,,d_2\Big\}.\Big\{-3456\,(3216\zeta_3-6960\zeta_5+2509)\,, \\
&\quad -144\left(48384\zeta_3^2-67600\zeta_3+1584\zeta_4+302400\zeta_5-86400\zeta_6-135571\right), \\
&\quad 8\left(1804032\zeta_3^2+1658520\zeta_3-2514888\zeta_4+3214080\zeta_5-777600\zeta_6-476417\right), \\
&\quad -18\left(382464\zeta_3^2+271448\zeta_3-865800\zeta_4+588576\zeta_5+316800\zeta_6-1524019\right), \\
&\quad 1658880\,(16\zeta_3-40\zeta_5+13)\,,13824\left(-288\zeta_3^2+4715\zeta_3-900\zeta_4+820\zeta_5-2373\right), \\
&\quad -27648\left(54\zeta_3^2-2354\zeta_3+360\zeta_4-295\zeta_5+225\zeta_6+230\right)\Big\}
\end{aligned}
$$

5-loop γ_3: Feynman gauge

$$
\begin{aligned}
\gamma_{341} \quad = \quad & \Big\{c_f^4, c_f^3, c_f^2, c_f, 1, c_f d_2, d_3, d_2\Big\}.\Big\{-41472\,(768\zeta_3+4157)\,, 82944\,(6164\zeta_3+1340\zeta_5-10080\zeta_7+11277)\,,\\
& -768\left(275400\zeta_3^2+1189209\zeta_3-195345\zeta_4-1942380\zeta_5+688500\zeta_6-1088640\zeta_7+2208371\right),\\
& 1152\left(168696\zeta_3^2-266283\zeta_3+385920\zeta_4-1119240\zeta_5+229500\zeta_6-300258\zeta_7+1139437\right),\\
& -8\left(-1137456\zeta_3^2-88182810\zeta_3+66057714\zeta_4+87060456\zeta_5-41007600\zeta_6-73764432\zeta_7+124662829\right),\\
& 331776\left(216\zeta_3^2+386\zeta_3+895\zeta_5+357\zeta_7-236\right),\\
& 1728\left(17760\zeta_3^2-232502\zeta_3+342\zeta_4+119960\zeta_5+80400\zeta_6-198198\zeta_7+11659\right),\\
& 3456\left(-61272\zeta_3^2-735952\zeta_3+150480\zeta_4+249580\zeta_5+76500\zeta_6+52479\zeta_7+77920\right)\Big\}\\
\gamma_{340} \quad = \quad & \Big\{1, c_f d_2, d_2, d_3\Big\}.\Big\{-32\,(21630996\zeta_3^2+116865396\zeta_3-57883140\zeta_4-484699320\zeta_5+115836750\zeta_6\\
& +272400975\zeta_7-112182361)\,, 331776\,(216\zeta_3^2+386\zeta_3+895\zeta_5+357\zeta_7-236)\,, 3456\,(-61272\zeta_3^2\\
& -735952\zeta_3+150480\zeta_4+249580\zeta_5+76500\zeta_6+52479\zeta_7+77920)\,,\\
& 1728\left(17760\zeta_3^2-232502\zeta_3+342\zeta_4+119960\zeta_5+80400\zeta_6-198198\zeta_7+11659\right)\Big\}
\end{aligned}
$$

β function

$$\partial_{\ln \mu^2} a = -a\Big[\varepsilon - \beta\Big] = -a\Big[\varepsilon + b_0\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\Big]$$

$$\begin{aligned}
3^1\, b_0 &= \Big[-4\Big] n_f + 11\ , \\
3^2\, b_1 &= \Big[-36 c_f - 60\Big] n_f + 102\ , \\
3^3\, b_2 &= \Big[132 c_f + 158\Big] n_f^2 + \Big[54 c_f^2 - 615 c_f - 1415\Big] n_f + 2857/2\ , \\
3^5\, b_3 &= \Big[1232 c_f + 424\Big] n_f^3 + 432(132\zeta_3 - 5) d_3 + (150653/2 - 1188\zeta_3) + \\
&\quad \Big[72(169 - 264\zeta_3) c_f^2 + 64(268 + 189\zeta_3) c_f + 1728(24\zeta_3 - 11) d_1 \\
&\quad + 6(3965 + 1008\zeta_3)\Big] n_f^2 + \Big[11178 c_f^3 + 36(264\zeta_3 - 1051) c_f^2 \\
&\quad + (7073 - 17712\zeta_3) c_f + 3456(4 - 39\zeta_3) d_2 + 3(3672\zeta_3 - 39143)\Big] n_f ,
\end{aligned}$$

5-loop β function

$$
\begin{aligned}
3^5\, b_4 &= b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}\ , \\
b_{44} &= \Big\{c_f, 1\Big\}.\Big\{ -8(107+144\zeta_3), 4(229-480\zeta_3)\Big\}\ , \\
b_{43} &= \Big\{c_f^2, c_f, d_1, 1\Big\}.\Big\{ -6(4961-11424\zeta_3+4752\zeta_4), -48(46+1065\zeta_3-378\zeta_4), \\
&\quad 1728(55-123\zeta_3+36\zeta_4+60\zeta_5), -3(6231+9736\zeta_3-3024\zeta_4-2880\zeta_5)\Big\}\ , \\
b_{42} &= \Big\{c_f^3, c_f^2, c_f d_1, c_f, d_2, d_1, 1\Big\}.\Big\{ -54(2509+3216\zeta_3-6960\zeta_5), \\
&\quad 9(94749/2-28628\zeta_3+10296\zeta_4-39600\zeta_5), 25920(13+16\zeta_3-40\zeta_5), \\
&\quad 3(5701/2+79356\zeta_3-25488\zeta_4+43200\zeta_5), -864(115-1255\zeta_3+234\zeta_4+40\zeta_5), \\
&\quad -432(1347-2521\zeta_3+396\zeta_4-140\zeta_5), 843067/2+166014\zeta_3-8424\zeta_4-178200\zeta_5\Big\}\ , \\
b_{41} &= \Big\{c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_3, d_2, 1\Big\}.\Big\{ -81(4157/2+384\zeta_3), 81(11151+5696\zeta_3-7480\zeta_5), \\
&\quad -3(548732+151743\zeta_3+13068\zeta_4-346140\zeta_5), -25920(3-4\zeta_3-20\zeta_5), \\
&\quad 8141995/8+35478\zeta_3+73062\zeta_4-706320\zeta_5, 216(113-2594\zeta_3+396\zeta_4+500\zeta_5), \\
&\quad 216(1414-15967\zeta_3+2574\zeta_4+8440\zeta_5), -5048959/4+31515\zeta_3-47223\zeta_4+298890\zeta_5\Big\}\ , \\
b_{40} &= \Big\{d_3, 1\Big\}.\Big\{ -162(257-9358\zeta_3+1452\zeta_4+7700\zeta_5), \\
&\quad 8296235/16-4890\zeta_3+9801\zeta_4/2-28215\zeta_5\Big\}\ .
\end{aligned}
$$

all results in full agreement with the literature

[Baikov,Chetyrkin,Kühn; Herzog,Ruijl,Ueda,Vermaseren]

Conclusions

- Conversion between the $\overline{\text{MS}}$ scheme and the on-shell scheme and various threshold mass schemes calculated at NNNLO
- Presented full gauge dependence up to four loops for QCD renormalization constants
- Presented the five-loop results for QCD renormalization constants (in Feynman gauge) including the β function
- All results are available for a general gauge group
- Full agreement with available results in the literature
- Completely independent calculation