6-loop $\phi^4$ theory in $4 - 2\varepsilon$ dimensions

Erik Panzer

All Souls College (Oxford)

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Methods and Applications, UPMC Paris

joint work with M. V. Kompaniets

Minimally subtracted six loop renormalization of $O(n)$-symmetric $\phi^4$ theory and critical exponents [arXiv:1705.06483]
Near the lambda transition ($T_\lambda \approx 2.2K$), the specific heat

$$C_p = \frac{A^\pm}{\alpha} |t|^{-\alpha} \left(1 + a_c^\pm |t|^\theta + b_c^\pm |t|^{2\theta} + \cdots \right) + B^\pm \quad \text{(for } T \gtrsim T_\lambda)$$

shows a power-law behaviour ($t = 1 - T / T_\lambda$).

$\Rightarrow \alpha = -0.0127(3)$
Near a phase transition at $T \to T_c$, a physical system can be described by power laws in terms of the reduced temperature $t = 1 - T/T_c$:

$$C_p \propto |t|^{-\alpha},$$
$$\xi \propto |t|^{-\nu} \quad \text{(correlation length)},$$
$$\chi \propto |t|^{-\gamma},$$
$$\langle \psi(0)\psi(r) \rangle \propto r^{2-d-\eta} \quad \text{(at } T = T_c).$$

Only two of these critical exponents are independent (scaling relations):

$$D\nu = 2 - \alpha, \quad \gamma = \nu(2 - \eta), \quad \alpha + 2\beta + \gamma = 2, \quad \beta\delta = \beta + \gamma.$$

**Universality**

Critical exponents depend only on:
- dimension $D$
- internal symmetry group, e.g. $O(n)$
Some $O(n)$ universality classes

$O(0)$ **self-avoiding walks**: diluted polymers

$O(1)$ **Ising model**: liquid-vapor transition, uniaxial magnets

$O(2)$ **XY universality class**: $\lambda$-transition of $^4$He, plane magnets

$O(3)$ **Heisenberg universality class**: isotropic magnets

---

**Onsager’s solution from 1944**

Exact solution of the Ising model in $D = 2$ dimensions:

$$\alpha = 0, \quad \beta = 1/8, \quad \nu = 1, \quad \eta = 1/4.$$
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So far, no exact solutions in $D = 3$ are known. Approximation methods:

1. lattice: Monte Carlo simulation, high temperature series
2. conformal bootstrap (recently: very high accuracy for $n = 1$)
3. RG ($\phi^4$ theory): in $D = 3$ dimensions
4. RG ($\phi^4$ theory): in $D = 4 - 2\varepsilon$ dimensions ($\varepsilon$-expansion) $\iff$ this talk
Consider scalar fields \( \phi = (\phi_1, \ldots, \phi_n) \) with \( O(n) \) symmetric interaction \( \phi^4 := (\phi^2)^2 \). The renormalized Lagrangian in \( D = 4 - 2\varepsilon \) dimensions is

\[
\mathcal{L} = \frac{1}{2} m^2 Z_1 \phi^2 + \frac{1}{2} Z_2 (\partial \phi)^2 + \frac{16}{4!} Z_4 g \mu^{2\varepsilon} \phi^4.
\]

The \( Z \)-factors relate the renormalized \((\phi, m, g)\) to the bare \((\phi_0, m_0, g_0)\) via

\[
Z_\phi = \frac{\phi_0}{\phi} = \sqrt{Z_2}, \quad Z_{m^2} = \frac{m_0^2}{m^2} = \frac{Z_1}{Z_2} \quad \text{and} \quad Z_g = \frac{g_0}{\mu^{2\varepsilon} g} = \frac{Z_4}{Z_2^2}.
\]

**Definition (RG functions: \( \beta \) and anomalous dimensions)**

\[
\beta(g) := \mu \left. \frac{\partial g}{\partial \mu} \right|_{g_0} \quad \gamma_{m^2}(g) := - \mu \left. \frac{\partial \log m^2}{\partial \mu} \right|_{m_0} \quad \gamma_\phi(g) := - \mu \left. \frac{\partial \log \phi}{\partial \mu} \right|_{\phi_0}
\]
RG equation

\[
\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - k \gamma \phi - \gamma m^2 m^2 \frac{\partial}{\partial m^2} \right] \Gamma_R^{(k)} (\vec{p}_1, \ldots, \vec{p}_k; m, g, \mu) = 0
\]

Near an IR-stable fixed point \( g_\star \), that is

\[ \beta(g_\star) = 0 \quad \text{and} \quad \beta'(g_\star) > 0, \]

the RG equation is solved by power laws and the critical exponents are

\[ \frac{1}{\nu} = 2 + \gamma m^2(g_\star), \quad \eta = 2 \gamma \phi(g_\star) \quad \text{and} \quad \omega = \beta'(g_\star). \]

(scheme independent)

Recall specific heat near \( \lambda \)-transition of \(^4\text{He}\)

\[ C_p = \frac{A^\pm}{\alpha} |t|^{-\alpha} \left( 1 + a_c^\pm |t|^{\theta} + b_c^\pm |t|^{2\theta} + \cdots \right) + B^\pm \quad \text{(for} \; T \geq T_\lambda) \]

The correction to scaling is determined by \( \theta = \omega \nu \approx 0.529 \).
DimReg and minimal subtraction (MS)

In MS, the $Z$-factors depend only on $\varepsilon$ and $g$ and admit expansions

$$Z_i = Z_i(g, \varepsilon) = 1 + \sum_{k=1}^{\infty} \frac{Z_{i,k}(g)}{\varepsilon^k}.$$ 

From their residues one can read off the RG functions:

$$\beta(g, \varepsilon) = -2\varepsilon g + 2g^2 \frac{\partial Z_{g,1}(g)}{\partial g}$$

and

$$\gamma_i(g) = -2g \frac{\partial Z_{i,1}(g)}{\partial g} \quad (i = m^2, \phi).$$

The critical coupling $g_\star = g_\star(\varepsilon)$ vanishes at $D = 4$ ($\varepsilon = 0$).

$\varepsilon$-expansions of critical exponents (formal power series)

$$\beta(g_\star(\varepsilon), \varepsilon) = 0,$$

$$\eta(\varepsilon) = 2\gamma_\phi(g_\star(\varepsilon))$$

$$\omega(\varepsilon) = \beta'(g_\star(\varepsilon)),$$

$$1/\nu(\varepsilon) = 2 + \gamma_{m^2}(g_\star(\varepsilon)).$$
In MS, the $Z$-factors are determined by the projection on poles

$$\mathcal{K} \left( \sum_k c_k \varepsilon^k \right) := \sum_{k<0} c_k \varepsilon^k$$

after subtraction of UV subdivergences using the $\mathcal{R}'$ operation:

$$Z_1 = 1 + \partial_{m^2} \mathcal{K} \mathcal{R}' \Gamma^{(2)}(p, m^2, g, \mu),$$

$$Z_2 = 1 + \partial_{p^2} \mathcal{K} \mathcal{R}' \Gamma^{(2)}(p, m^2, g, \mu) \quad \text{and}$$

$$Z_4 = 1 + \mathcal{K} \mathcal{R}' \Gamma^{(4)}(p, m^2, g, \mu)/g.$$

Summary of this method

1. Compute $\varepsilon$-expansions of dimensionally regulated Feynman integrals of $O(n)$-symmetric $\phi^4$ theory.
2. Combine them with $\mathcal{R}'$ and $\mathcal{K}$ to obtain $Z$-factors.
3. Deduce RG functions and critical exponents.
4. By universality, these should describe many different physical systems.
computational techniques
infrared rearrangement (IRR)

First note that $Z$-factors do not depend on $m^2$. Using

$$\frac{\partial}{\partial m^2} \frac{1}{k^2 + m^2} = -\frac{1}{k^2 + m^2} \frac{1}{k^2 + m^2},$$

$Z_2$ can be expressed in terms of a subset of $\Gamma^{(4)}$-graphs.

$\Rightarrow$ we can set all masses to zero

More generally, if a graph $G$ is superficially log. divergent and primitive (no subdivergences), then its residue is independent of kinematics:

$$\Phi(G; \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) = \frac{P(G)}{\text{loops}(G)\varepsilon} + O(\varepsilon^0)$$

**Example**

$$\Phi\left(\begin{array}{c} \vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4 \end{array}\right) = \frac{2\zeta_3}{\varepsilon} + O(\varepsilon^0) \quad \text{where} \quad \zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}$$
Some traditional methods

- use IRR to reduce all $\mathcal{K}\mathcal{R}'\Phi(G)$ to massless propagators ($\rho$-integrals):
  
  \[ G = \begin{tikzpicture} [baseline=(current bounding box.center)]
  \draw [fill=white] (0,0) circle (0.1cm);
  \draw [fill=white] (1,1) circle (0.1cm);
  \draw [fill=white] (-1,1) circle (0.1cm);
  \draw [thick] (0,0) -- (1,1);
  \draw [thick] (0,0) -- (-1,1);
  \end{tikzpicture} \quad \mapsto \quad \begin{tikzpicture} [baseline=(current bounding box.center)]
  \draw [fill=white] (0,0) circle (0.1cm);
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  \draw [thick] (0,0) -- (1,1);
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  \end{tikzpicture} = G_{\text{IR-safe}}^{1-s}, \text{ but not } G_{\text{IR-unsafe}}^{1-s} = \begin{tikzpicture} [baseline=(current bounding box.center)]
  \draw [fill=white] (0,0) circle (0.1cm);
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  \end{tikzpicture} \]

- $\mathcal{R}^*$ extends this by allowing for IR-divergences ($\Rightarrow$ trivializes a loop):
  
  \[ \begin{tikzpicture} [baseline=(current bounding box.center)]
  \draw [fill=white] (0,0) circle (0.1cm);
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  \end{tikzpicture} + \text{IR counter terms} \]

- factorization of 1-scale subgraphs:
  
  \[ \begin{tikzpicture} [baseline=(current bounding box.center)]
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  \end{tikzpicture} \right)^2 \begin{tikzpicture} [baseline=(current bounding box.center)]
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  \end{tikzpicture} \]

- IBP: only up to 4 loops!

Automatized and implemented (open source) [Batkovich & Kompaniets ’14].
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- $\mathcal{R}^*$ extends this by allowing for IR-divergences ($\Rightarrow$ trivializes a loop):

  $\mapsto \quad + \text{IR counter terms}$

- factorization of 1-scale subgraphs:

  $= \begin{pmatrix} \varepsilon \end{pmatrix}^2$

- IBP: only up to 4 loops!

Automatized and implemented (open source) [Batkovich & Kompaniets ’14].
irreducible (not 4-loop reducible) 6-loop $\phi^4$ integrals
Some history

4 loops
- critical exponents [Brezin, LeGuillou & Zinn-Justin ’74], [Kazakov, Tarasov & Vladimirov ’79]
- 3-loop propagators [Chetyrkin & Tkachov ’81]

5 loops
- $\gamma_{\phi}$ [Chetyrkin, Kataev, Tkachov ’81]
- $\beta$ [Chetyrkin, Gorishny, Larin, Tkachov ’83 ’86, Kazakov ’83]
- corrections [Kleinert, Neu, Schulte-Frohlinde, Chetyrkin, Larin ’91, ’93]
- numeric checks [Adzhemyan, Kompaniets ’14]
- 4-loop propagators [Baikov & Chetyrkin, Smirnov & Tentyukov ’10]
  with arbitrary indices [Panzer ’13]

6 loops
- primitives [Broadhurst ’85], 5-loop propagator [Broadhurst ’93]
- $\gamma_{\phi}$ [Batkovich, Kompaniets, Chetyrkin ’16]
- $\beta$ and $\gamma_{m^2}$ [Kompaniets & Panzer ’16]
- independent computation [Schnetz ’16]

7 loops
- primitives [Broadhurst & Kreimer ’95], [Schnetz ’10], [Panzer ’14]
- $\gamma_{\phi}$ [Schnetz ’16]
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7 loops

- primitives [Broadhurst & Kreimer '95], [Schnetz '10], [Panzer '14]
- $\gamma$ [Schnetz '16], also $\beta$ & $\gamma_m^2$ [Schnetz '17]
New methods

1. Parametric integration with hyperlogarithms
2. Resolution of singularities
   - via IBP [Panzer '14], [von Manteuffel, Panzer & Schabinger '15]
   - primitive linear combinations
   - one-scale scheme [Brown & Kreimer '13]
3. Graphical functions [Schnetz '14]
   - generalized single-valued hyperlogarithms [Schnetz]
   - combined with parametric integration [Golz, Panzer & Schnetz '16]

We do not use any IBP reductions and compute all Feynman integrals. This is feasible because $\phi^4$ theory has only very few graphs:

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Parametric integration

The $\alpha$-representation of $\mathcal{P}(G)$ for a primitive graph is

$$\mathcal{P}(G) = \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_{N-1} \frac{1}{\psi^2|_{\alpha_N=1}}$$

where the Kirchhoff/graph/first Symanzik polynomial is

$$\psi = \mathcal{U} = \sum_{T\text{ spanning tree }} \prod_{e \not\in T} \alpha_e.$$

For linearly reducible graphs $G$, this integral can be computed exactly in terms of polylogarithms [HyperInt] (open source).

```plaintext
> read "HyperInt.mpl":
> E := [[1,2],[2,3],[3,1],[1,4],[2,4],[3,4]]:
> hyperInt(1/psi^2, [x[1],x[2],x[3],x[4],x[5]]):
> fibrationBasis(%);

6ζ_3
```
- check for linear reducibility available (HyperInt)
- fulfilled for all but one $\phi^4$ graph up to $\leq 6$ loops
- applies also to some non-propagator integrals
- integration works in $2n - 2\varepsilon$ dimensions
- $\varepsilon$-dependent propagator exponents allowed

\[
\mathcal{P} = \frac{92943}{160} \zeta_{11} + \frac{3381}{20} \left( \zeta_{3,5,3} - 3\zeta_3 \zeta_5 \right) - \frac{1155}{4} \zeta_3^2 \zeta_5
\]

\[
+ 896 \zeta_3 \left( \frac{27}{80} \zeta_{3,5} + \frac{45}{64} \zeta_3 \zeta_5 - \frac{261}{320} \zeta_8 \right)
\]

Survey of primitive periods up to 11 loops

*The Galois coaction on $\phi^4$ periods (w. Oliver Schnetz)*
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Survey of primitive periods up to 11 loops

The Galois coaction on $\phi^4$ periods (w. Oliver Schnetz)
How to deal with divergences?

- Sector decomposition $\iff$ very tough & only numeric (at 6 loops)
- IBP to finite master integrals $\iff$ no IBP at 6 loops
- primitive linear combinations $\iff$ non-trivial to automate
- one-scale BPHZ
Renormalization of subdivergences:

\[ \Phi_R = \Phi - \Phi^0 \left( \frac{1}{G} \right) \Phi \left( \frac{1}{G} \right) - \Phi^0 \left( \frac{1}{G} \right) \Phi \left( \frac{1}{G} \right) - \Phi^0 \left( \frac{1}{G} \right) \Phi \left( \frac{1}{G} \right) + 2\Phi^0 \left( \frac{1}{G} \right) \Phi \left( \frac{1}{G} \right) \Phi \left( \frac{1}{G} \right) \]

BPHZ-like scheme

\[ \Phi^0(G) := \Phi(G) \text{ at a fixed renormalization point } (p^0_1, \cdots, p^0_4, m_0) \]

Theorem (Renormalization under the integral sign, Weinberg '60)

The BPHZ-subtracted integrand is integrable. (This is false in MS!)
one-scale renormalization scheme

BPHZ renormalization of log. UV subdivergences via forest formula:

\[ \Phi_R(G) = \sum_{F \in \mathcal{F}(G)} (-1)^F \prod_{\gamma \in G} \Phi^0(\gamma) \Phi(G/\gamma) \]

Idea [Brown & Kreimer ’13]: Choose \( \Phi^0(\gamma) := \Phi(\gamma^0)|_{p^2=1} \) to be 1-scale!

Example

\[ G = \begin{array}{c}
\includegraphics[scale=0.5]{example1}
\end{array} \quad \supset \quad \gamma = \begin{array}{c}
\includegraphics[scale=0.5]{example2}
\end{array} \quad \Rightarrow \quad \gamma^0 = \begin{array}{c}
\includegraphics[scale=0.5]{example3}
\end{array} \]

- \( \Phi_R(G) \) is a convergent integral at \( \varepsilon = 0 \)
  \( \Rightarrow \) HyperInt (\( \varepsilon \)-expansion under the integral sign)
- \( \Phi(G) = \Phi_R(G) + \sum \) products of lower-loop \( p \)-integrals
- easy to implement
1. compute forest formula & choose IR-safe one-scale structures $\gamma^0$
2. integrate the (convergent) $\partial p^2 \Phi_R(G) (\Rightarrow \text{HyperInt})$
3. solve for $\Phi(G)$, using products of lower-loop integrals

\[ \mathcal{K}\mathcal{R}' \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) = -\frac{\zeta_3}{3\varepsilon^4} + \left( \frac{5}{3} \zeta_3 + \frac{\pi^4}{180} \right) \frac{1}{\varepsilon^3} \\
\quad - \left( \frac{9}{2} \zeta_3 + \frac{7\pi^4}{360} - \frac{23}{6} \zeta_5 \right) \frac{1}{\varepsilon^2} \\
\quad + \left( \frac{9}{2} \zeta_3 + \frac{7\pi^4}{360} - \frac{161}{30} \zeta_5 + \frac{7}{10} \zeta_3^2 - \frac{2\pi^6}{945} \right) \frac{1}{\varepsilon} \]
results
\[ \gamma_{\phi}^{\text{MS}}(g) = \frac{n + 2}{36} g^2 - \frac{(n + 8)(n + 2)}{432} g^3 - \frac{5(n^2 - 18n - 100)(n + 2)}{5184} g^4 \]
\[ - \left[ 1152(5n + 22)\zeta_4 - 48(n^3 - 6n^2 + 64n + 184)\zeta_3 \right. \]
\[ + (39n^3 + 296n^2 + 22752n + 77056) \left. \right] \frac{(n + 2)g^5}{186624} \]
\[ - \left[ 512(2n^2 + 55n + 186)\zeta_3^2 - 6400(2n^2 + 55n + 186)\zeta_6 \right. \]
\[ + 4736(n + 8)(5n + 22)\zeta_5 \]
\[ - 48(n^4 + 2n^3 + 328n^2 + 4496n + 12912)\zeta_4 \]
\[ + 16(n^4 - 936n^2 - 4368n - 18592)\zeta_3 \]
\[ + (29n^4 + 794n^3 - 30184n^2 - 549104n - 1410544) \right] \frac{(n + 2)g^6}{746496} \]
\[ + \mathcal{O}\left( g^7 \right) \]

Check: large \( n \)-expansions [Vasilev, Pismak & Honkonen ’81]
for \( \beta \): [Broadhurst, Gracey & Kreimer ’97]
Result \(( N = 1 \), \( D = 4 - 2\varepsilon \))

\[
\beta_{MS}^\ell (g) = -2\varepsilon g + 3g^2 - \frac{17}{3}g^3 + \left( \frac{145}{8} + 12\zeta_3 \right) g^4 \\
- \left( 120\zeta_5 - 18\zeta_4 + 78\zeta_3 + \frac{3499}{48} \right) g^5 \\
+ \left( 1323\zeta_7 + 45\zeta_3^2 - \frac{675}{2}\zeta_6 + 987\zeta_5 - \frac{1189}{8}\zeta_4 + \frac{7965}{16}\zeta_3 + \frac{764621}{2304} \right) g^6 \\
- \left( \frac{46112}{3}\zeta_9 + 768\zeta_3^3 + \frac{51984}{25}\zeta_3,5 - \frac{264543}{25}\zeta_8 + 4704\zeta_3\zeta_5 \\
+ \frac{63627}{5}\zeta_7 - 162\zeta_3\zeta_4 + \frac{8678}{5}\zeta_3^2 - \frac{6691}{2}\zeta_6 + \frac{63723}{10}\zeta_5 \\
- \frac{16989}{16}\zeta_4 + \frac{779603}{240}\zeta_3 + \frac{18841427}{11520} \right) g^7 \\
+ \mathcal{O} \left( g^8 \right)
\]

Numerical values: \( \zeta_{3,5} = \sum_{1 \leq n < m} \frac{1}{n^3 m^5} \approx 0.037707673 \)

\[
\approx -2\varepsilon g + 3g^2 - 5.7g^3 + 32.6g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + \mathcal{O} \left( g^8 \right)
\]
asymptotics
Let $\beta_{\text{MS}}(g) = \sum_k \beta_{\text{MS}}^k (-g)^k$.

**Asymptotics of the perturbation series**

According to [McKane, Wallace & Bonfim ’84],

$$\beta_{\text{MS}}^k \sim \bar{\beta}_k := k! \cdot k^{3+n/2} \cdot C_\beta \quad \text{as } k \to \infty$$

where $C_\beta$ is a constant that only depends on $n$:

$$C_\beta = \frac{36 \cdot 3^{(n+1)/2}}{\pi \Gamma(2 + n/2) A^{2n+4}} \exp \left[ -\frac{3}{2} - \frac{n+8}{3} \left( \gamma_E + \frac{3}{4} \right) \right].$$

$\gamma_E \approx 0.577$ (Euler-Mascheroni) and $A \approx 1.282$ (Glaisher-Kinkelin)
\[ \frac{\beta_k}{k! \cdot k^{3+n/2} \cdot C_\beta} \]
<table>
<thead>
<tr>
<th>loop order $\ell$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{MS}^{\ell+1}/\beta_{l+1}^{MS}$ in %</td>
<td>548</td>
<td>83.5</td>
<td>43.8</td>
<td>33.5</td>
<td>30.9</td>
<td>31.4</td>
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<td>32.5</td>
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<td>$\beta_{l+1}^{prim}/\beta_{l+1}^{MS}$ in %</td>
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<td>0</td>
<td>44.3</td>
<td>45.8</td>
<td>59.6</td>
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<td>4-point graphs</td>
<td>1</td>
<td>2</td>
<td>8</td>
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<td>627</td>
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<tr>
<td>primitives</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10</td>
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</table>

$$C_\beta = \frac{36 \cdot 3^{(n+1)/2}}{\pi \Gamma(2 + n/2) A^{2n+4}} \exp \left[ -\frac{3}{2} - \frac{n+8}{3} \left( \gamma_E + \frac{3}{4} \right) \right]$$
<table>
<thead>
<tr>
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<th>second zero</th>
<th>third zero</th>
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<tbody>
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<tr>
<td>2</td>
<td>-4.67</td>
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<tr>
<td>3</td>
<td>-4.025</td>
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<td>4</td>
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<td>-12.1</td>
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<td>5</td>
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<tr>
<td>6</td>
<td>-4.00044</td>
<td>-7.52</td>
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<tr>
<td>6</td>
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<td>-7.22</td>
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<td>7</td>
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<td>-3.99994</td>
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<td>9</td>
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<tr>
<td>11</td>
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<td>-8.21</td>
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</tbody>
</table>
resummation
The $\varepsilon$-expansion $f(\varepsilon) = \sum_{n \geq 0} f_n \varepsilon^n$ of crit. exponents is divergent:

$$f_n \sim C n! a^n n^{b_0} \quad [\text{McKane, Wallace & Bonfim '84}]$$

Borel-resummation after [Le Guillou & J. Zinn-Justin '85]:

$$f(\varepsilon) = \int_0^\infty x^{b-1} \tilde{f}(x) e^{-x/\varepsilon} \, dx \quad \text{with} \quad \tilde{f}_n = \frac{f_n}{\Gamma(n + b)}$$

Conformal mapping (analytic continuation):

$$\tilde{f}(x) = \left( \frac{x}{w} \right)^\lambda \left( a_0 + a_1 w + \ldots + a_\ell w^\ell \right) \quad \text{where} \quad w(x) = \frac{\sqrt{1 + x - 1}}{\sqrt{1 + x + 1}}$$

Homographic transformation: Re-expand in $\varepsilon'$ given by

$$\varepsilon = \frac{\varepsilon'}{1 + q \varepsilon'}$$
$$\eta(n = 1) \approx 0.07407\varepsilon^2 + 0.1495\varepsilon^3 - 0.1333\varepsilon^4 + 0.8210\varepsilon^5 - 5.2015\varepsilon^6 + \mathcal{O}(\varepsilon^7)$$
\[ \eta(n = 1) \approx 0.07407\varepsilon^2 + 0.1495\varepsilon^3 - 0.1333\varepsilon^4 + 0.8210\varepsilon^5 - 5.2015\varepsilon^6 + O(\varepsilon^7) \]
### 3d critical exponents

<table>
<thead>
<tr>
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<th>$n = 0$</th>
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<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
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<tbody>
<tr>
<td>$\eta^6$</td>
<td>0.031043(3)</td>
<td>0.036298(2)</td>
<td>0.0381(2)</td>
<td>0.0378(3)</td>
<td>0.0360(3)</td>
</tr>
<tr>
<td>$\eta^5$</td>
<td>0.0310(7)</td>
<td>0.0362(6)</td>
<td>0.0380(6)</td>
<td>0.0378(5)</td>
<td>0.0366(4)</td>
</tr>
<tr>
<td>G/ZJ</td>
<td>0.0300(50)</td>
<td>0.0360(50)</td>
<td>0.0380(50)</td>
<td>0.0375(45)</td>
<td>0.036(4)</td>
</tr>
<tr>
<td>$\nu^6$</td>
<td>0.5875970(4)</td>
<td>0.629971(4)</td>
<td>0.6717(1)</td>
<td>0.7112(5)</td>
<td>0.7477(8)</td>
</tr>
<tr>
<td>$\nu^5$</td>
<td>0.5874(3)</td>
<td>0.6292(5)</td>
<td>0.6690(10)</td>
<td>0.7059(20)</td>
<td>0.7397(35)</td>
</tr>
<tr>
<td>G/ZJ</td>
<td>0.5875(25)</td>
<td>0.6290(25)</td>
<td>0.6680(35)</td>
<td>0.7045(55)</td>
<td>0.737(8)</td>
</tr>
<tr>
<td>$\omega^6$</td>
<td>0.904(5)</td>
<td>0.830(2)</td>
<td>0.811(10)</td>
<td>0.791(22)</td>
<td>0.817(30)</td>
</tr>
<tr>
<td>$\omega^5$</td>
<td>0.841(13)</td>
<td>0.820(7)</td>
<td>0.804(3)</td>
<td>0.795(7)</td>
<td>0.794(9)</td>
</tr>
<tr>
<td>G/ZJ</td>
<td>0.828(23)</td>
<td>0.814(18)</td>
<td>0.802(18)</td>
<td>0.794(18)</td>
<td>0.795(30)</td>
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### 2d critical exponents

<table>
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<th>$n = -1$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\eta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>0.15</td>
<td>0.208333...</td>
<td>0.25</td>
</tr>
<tr>
<td>$\varepsilon^6$</td>
<td>0.130(17)</td>
<td>0.201(25)</td>
<td>0.237(27)</td>
</tr>
<tr>
<td>$\varepsilon^5$</td>
<td>0.137(23)</td>
<td>0.215(35)</td>
<td>0.249(38)</td>
</tr>
<tr>
<td>LeG/ZJ</td>
<td>0.21(5)</td>
<td>0.26(5)</td>
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</tr>
<tr>
<td>$\nu$</td>
<td></td>
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</tr>
<tr>
<td>N</td>
<td>0.625</td>
<td>0.75</td>
<td>1</td>
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<tr>
<td>$\varepsilon^6$</td>
<td>0.6036(23)</td>
<td>0.741(4)</td>
<td>0.952(14)</td>
</tr>
<tr>
<td>$\varepsilon^5$</td>
<td>0.6025(27)</td>
<td>0.747(20)</td>
<td>0.944(48)</td>
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<tr>
<td>LeG/ZJ</td>
<td>0.76(3)</td>
<td>0.99(4)</td>
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<tr>
<td>$\omega$</td>
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<tr>
<td>$\varepsilon^6$</td>
<td>1.95(28)</td>
<td>1.90(25)</td>
<td>1.71(9)</td>
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<tr>
<td>$\varepsilon^5$</td>
<td>1.88(30)</td>
<td>1.83(25)</td>
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<tr>
<td>LeG/ZJ</td>
<td>1.7(2)</td>
<td>1.6(2)</td>
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</tbody>
</table>
new tools for massless propagators
$\phi^4$ beta function at six loops
higher accuracy for critical exponents in $D = 3$
Thanks

Thank you for your attention!

- new tools for massless propagators
- $\phi^4$ beta function at six loops
- higher accuracy for critical exponents in $D = 3$

Stay tuned

tomorrow: $\phi^4$ at 7 loops, by Oliver Schnetz
Alternative method

Given a graph $G$, find a linear combination $X$ of graphs such that

1. $G - X$ is primitive (free of subdivergences) ($\Rightarrow$ HyperInt)
2. each term in $X$ factorizes (has a $\geq 1$ loop sub-$p$-integral) [Panzer '13]

Example

Both have the same subdivergence $\gamma$ and quotient $G/\gamma$:

- simple: just $p$-integrals, no renormalization
- not straightforward to automate