6-loop ϕ^4 theory in $4-2\varepsilon$ dimensions

Erik Panzer

All Souls College (Oxford)

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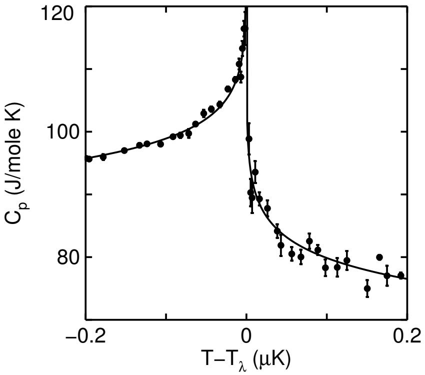
joint work with M. V. Kompaniets
Minimally subtracted six loop renormalization of $O(n)$-symmetric ϕ^4 theory and critical exponents [arXiv:1705.06483]

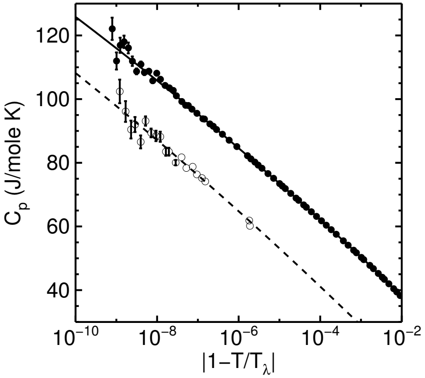
Outline

1. Motivation
2. Calculational techniques
3. Results

λ-transition of $^4\mathrm{He}$ (Columbia, October 1992)

Specific heat of liquid helium in zero gravity very near the lambda point [Lipa, Nissen, Stricker, Swanson & Chui '03]





Near the lambda transition ($T_\lambda \approx 2.2K$), the specific heat

$$C_p = \frac{A^\pm}{\alpha} |t|^{-\alpha} \left(1 + a_c^\pm |t|^\theta + b_c^\pm |t|^{2\theta} + \cdots \right) + B^\pm \quad (\text{for } T \gtrless T_\lambda)$$

shows a power-law behaviour ($t = 1 - T/T_\lambda$). $\Rightarrow \alpha = -0.0127(3)$

Near a phase transition at $T \to T_c$, a physical system can be described by power laws in terms of the reduced temperature $t = 1 - T/T_c$:

$$C_p \propto |t|^{-\alpha}, \qquad \xi \propto |t|^{-\nu} \text{ (correlation length)},$$
$$\chi \propto |t|^{-\gamma}, \qquad \langle \psi(0)\psi(r) \rangle \propto r^{2-d-\eta} \text{ (at } T = T_c\text{)}.$$

Only two of these critical exponents are independent (scaling relations):

$$D\nu = 2 - \alpha, \quad \gamma = \nu(2-\eta), \quad \alpha + 2\beta + \gamma = 2, \quad \beta\delta = \beta + \gamma.$$

Universality

Critical exponents depend only on:

- dimension D
- internal symmetry group, e. g. $O(n)$

Some $O(n)$ universality classes

$O(0)$ **self-avoiding walks**: diluted polymers

$O(1)$ **Ising model**: liquid-vapor transition, uniaxial magnets

$O(2)$ **XY universality class**: λ-transition of $^4\mathrm{He}$, plane magnets

$O(3)$ **Heisenberg universality class**: isotropic magnets

Onsager's solution from 1944

Exact solution of the Ising model in $D = 2$ dimensions:

$$\alpha = 0, \quad \beta = 1/8, \quad \nu = 1, \quad \eta = 1/4.$$

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So far, no exact solutions in $D = 3$ are known. Approximation methods:

1. lattice: Monte Carlo simulation, high temperature series
2. conformal bootstrap (recently: very high accuracy for $n = 1$)
3. RG (ϕ^4 theory): in $D = 3$ dimensions
4. RG (ϕ^4 theory): in $D = 4 - 2\varepsilon$ dimensions (ε-expansion) $\Leftarrow$ this talk

Consider scalar fields $\phi = (\phi_1, \ldots, \phi_n)$ with $O(n)$ symmetric interaction $\phi^4 := (\phi^2)^2$. The renormalized Lagrangian in $D = 4 - 2\varepsilon$ dimensions is

$$\mathscr{L} = \frac{1}{2} m^2 Z_1 \phi^2 + \frac{1}{2} Z_2 \, (\partial\phi)^2 + \frac{16\,\pi^2}{4!} Z_4 \, g \, \mu^{2\varepsilon} \, \phi^4.$$

The Z-factors relate the renormalized (ϕ, m, g) to the bare (ϕ_0, m_0, g_0) via

$$Z_\phi = \frac{\phi_0}{\phi} = \sqrt{Z_2}, \quad Z_{m^2} = \frac{m_0^2}{m^2} = \frac{Z_1}{Z_2} \quad \text{and} \quad Z_g = \frac{g_0}{\mu^{2\varepsilon} g} = \frac{Z_4}{Z_2^2}.$$

Definition (RG functions: β and anomalous dimensions)

$$\beta(g) := \mu \frac{\partial g}{\partial \mu}\bigg|_{g_0} \quad \gamma_{m^2}(g) := -\mu \frac{\partial \log m^2}{\partial \mu}\bigg|_{m_0} \quad \gamma_\phi(g) := -\mu \frac{\partial \log \phi}{\partial \mu}\bigg|_{\phi_0}$$

RG equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}-k\gamma_\phi-\gamma_{m^2}m^2\frac{\partial}{\partial m^2}\right]\Gamma_R^{(k)}\left(\vec{p_1},\ldots,\vec{p_k};m,g,\mu\right)=0$$

Near an IR-stable fixed point $g_\star$, that is

$$\beta(g_\star)=0 \quad \text{and} \quad \beta'(g_\star)>0,$$

the RG equation is solved by power laws and the critical exponents are

$$1/\nu=2+\gamma_{m^2}(g_\star), \quad \eta=2\gamma_\phi(g_\star) \quad \text{and} \quad \omega=\beta'(g_\star).$$

(scheme independent)

Recall specific heat near λ-transition of $^4\mathrm{He}$

$$C_p=\frac{A^\pm}{\alpha}|t|^{-\alpha}\left(1+a_c^\pm|t|^\theta+b_c^\pm|t|^{2\theta}+\cdots\right)+B^\pm \quad (\text{for } T\gtrless T_\lambda)$$

The correction to scaling is determined by $\theta=\omega\nu\approx 0.529$.

DimReg and minimal subtraction (MS)

In MS, the Z-factors depend only on ε and g and admit expansions

$$Z_i = Z_i(g,\varepsilon) = 1 + \sum_{k=1}^{\infty} \frac{Z_{i,k}(g)}{\varepsilon^k}.$$

From their residues one can read off the RG functions:

$$\beta(g,\varepsilon) = -2\varepsilon g + 2g^2 \frac{\partial Z_{g,1}(g)}{\partial g} \quad \text{and} \quad \gamma_i(g) = -2g \frac{\partial Z_{i,1}(g)}{\partial g} \quad (i = m^2, \phi).$$

The critical coupling $g_\star = g_\star(\varepsilon)$ vanishes at $D = 4$ ($\varepsilon = 0$).

ε-expansions of critical exponents (formal power series)

$$\beta(g_\star(\varepsilon),\varepsilon) = 0, \qquad \eta(\varepsilon) = 2\gamma_\phi(g_\star(\varepsilon))$$

$$\omega(\varepsilon) = \beta'(g_\star(\varepsilon)), \qquad 1/\nu(\varepsilon) = 2 + \gamma_{m^2}(g_\star(\varepsilon)).$$

In MS, the Z-factors are determined by the projection on poles

$$\mathcal{K}\left(\sum_k c_k \varepsilon^k\right) := \sum_{k<0} c_k \varepsilon^k$$

after subtraction of UV subdivergences using the $\mathcal{R}'$ operation:

$$Z_1 = 1 + \partial_{m^2} \mathcal{K}\mathcal{R}'\Gamma^{(2)}(p, m^2, g, \mu),$$
$$Z_2 = 1 + \partial_{p^2} \mathcal{K}\mathcal{R}'\Gamma^{(2)}(p, m^2, g, \mu) \quad \text{and}$$
$$Z_4 = 1 + \mathcal{K}\mathcal{R}'\Gamma^{(4)}(p, m^2, g, \mu)/g.$$

Summary of this method

1. Compute ε-expansions of dimensionally regulated Feynman integrals of $O(n)$-symmetric ϕ^4 theory.
2. Combine them with $\mathcal{R}'$ and $\mathcal{K}$ to obtain Z-factors.
3. Deduce RG functions and critical exponents.
4. By universality, these should describe many different physical systems.

computational techniques

infrared rearrangement (IRR)

First note that Z-factors do not depend on m^2. Using

$$\frac{\partial}{\partial m^2}\frac{1}{k^2+m^2} = -\frac{1}{k^2+m^2}\frac{1}{k^2+m^2},$$

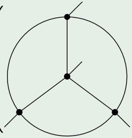
Z_2 can be expressed in terms of a subset of $\Gamma^{(4)}$-graphs.

$\Rightarrow$ we can set all masses to zero

More generally, if a graph G is superficially log. divergent and primitive (no subdivergences), then its residue is independent of kinematics:

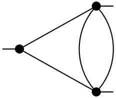
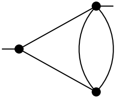
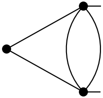
$$\Phi(G;\vec{p_1},\vec{p_2},\vec{p_3},\vec{p_4}) = \frac{\mathcal{P}(G)}{\text{loops}(G)\varepsilon} + \mathcal{O}\left(\varepsilon^0\right)$$

Example

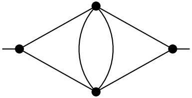
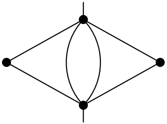
$\Phi\Big($  $;\vec{p_1},\vec{p_2},\vec{p_3},\vec{p_4}\Big) = \dfrac{2\zeta_3}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right)$ where $\zeta_n = \sum_{k=1}^{\infty}\dfrac{1}{k^n}$

Some traditional methods

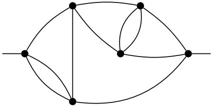
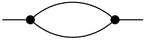
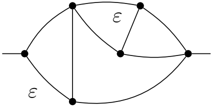
- use IRR to reduce all $\mathcal{K}\mathcal{R}'\Phi(G)$ to massless propagators (*p*-integrals):

$G =$  $\mapsto\ G_{\text{1-s}}^{\text{IR-safe}} =$ , but not $G_{\text{1-s}}^{\text{IR-unsafe}} =$ 

- $\mathcal{R}^*$ extends this by allowing for IR-divergences ($\Rightarrow$ trivializes a loop):

 $\mapsto$  $+$ IR counter terms

- factorization of 1-scale subgraphs:

 $= \Big($  $\Big)^2$ 

- IBP: only up to 4 loops!

Automatized and implemented (open source) [Batkovich & Kompaniets '14].

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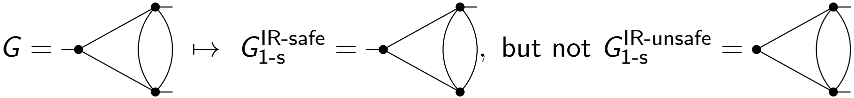

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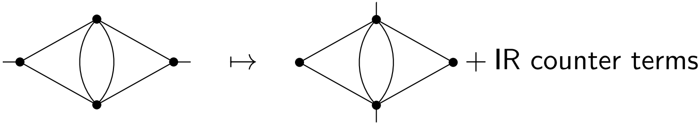

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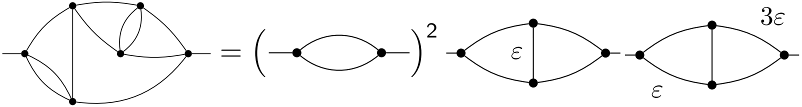

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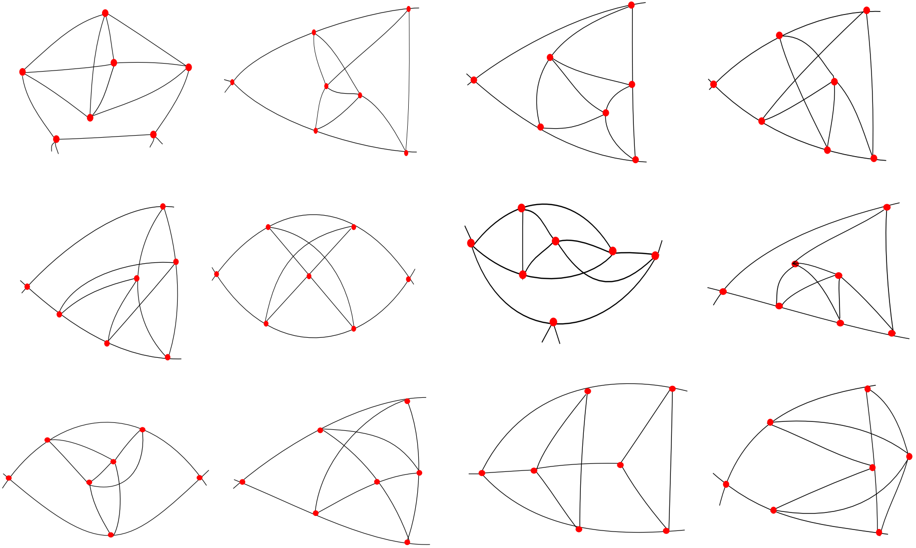
irreducible (not 4-loop reducible) 6-loop ϕ^4 integrals



Some history

4 loops

- critical exponents [Brezin, LeGuillou & Zinn-Justin '74], [Kazakov, Tarasov & Vladimirov '79]
- 3-loop propagators [Chetyrkin & Tkachov '81]

5 loops

- γ_ϕ [Chetyrkin, Kataev, Tkachov '81]
- β [Chetyrkin, Gorishny, Larin, Tkachov '83 '86, Kazakov '83]
- corrections [Kleinert, Neu, Schulte-Frohlinde, Chetyrkin, Larin '91, '93]
- numeric checks [Adzhemyan, Kompaniets '14]
- 4-loop propagators [Baikov & Chetyrkin, Smirnov & Tentyukov '10] with arbitrary indices [Panzer '13]

6 loops

- primitives [Broadhurst '85], 5-loop propagator [Broadhurst '93]
- γ_ϕ [Batkovich, Kompaniets, Chetyrkin '16]
- β and γ_{m^2} [Kompaniets & Panzer '16]
- independent computation [Schnetz '16]

7 loops

- primitives [Broadhurst & Kreimer '95], [Schnetz '10], [Panzer '14]
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7 loops

- primitives [Broadhurst & Kreimer '95], [Schnetz '10], [Panzer '14]
- γ_ϕ [Schnetz '16] , also β & γ_{m^2} [Schnetz '17]

New methods

1. Parametric integration with hyperlogarithms
2. Resolution of singularities
   - via IBP [Panzer '14], [von Manteuffel, Panzer & Schabinger '15]
   - primitive linear combinations
   - one-scale scheme [Brown & Kreimer '13]
3. Graphical functions [Schnetz '14]
   - generalized single-valued hyperlogarithms [Schnetz]
   - combined with parametric integration [Golz, Panzer & Schnetz '16]

We do not use any IBP reductions and compute all Feynman integrals.
This is feasible because ϕ^4 theory has only very few graphs:

# loops	1	2	3	4	5	6
# 4-point graphs	1	2	8	26	124	627

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Parametric integration

The α-representation of $\mathcal{P}(G)$ for a primitive graph is

$$\mathcal{P}(G) = \int_0^\infty \mathrm{d}\alpha_1 \cdots \int_0^\infty \mathrm{d}\alpha_{N-1} \frac{1}{\psi^2|_{\alpha_N=1}}$$

where the Kirchhoff/graph/first Symanzik polynomial is

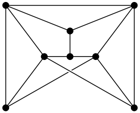
$$\psi = \mathcal{U} = \sum_{T \text{ spanning tree}} \prod_{e \notin T} \alpha_e.$$

For linearly reducible graphs G, this integral can be computed exactly in terms of polylogarithms [HyperInt] (open source).

```
> read "HyperInt.mpl":
> E := [[1,2],[2,3],[3,1],[1,4],[2,4],[3,4]]:
> psi := eval(graphPolynomial(E), x[6]=1):
> hyperInt(1/psi^2,[x[1],x[2],x[3],x[4],x[5]]):
> fibrationBasis(%);
```

$$6\zeta_3$$

- check for linear reducibility available (`HyperInt`)
- fulfilled for all but one ϕ^4 graph up to ≤ 6 loops
- applies also to some non-propagator integrals
- integration works in $2n - 2\varepsilon$ dimensions
- ε-dependent propagator exponents allowed

$\mathcal{P}$ ()

$$= \frac{92\,943}{160}\zeta_{11} + \frac{3381}{20}\left(\zeta_{3,5,3} - \zeta_3\zeta_{3,5}\right) - \frac{1155}{4}\zeta_3^2\zeta_5 + 896\zeta_3\left(\frac{27}{80}\zeta_{3,5} + \frac{45}{64}\zeta_3\zeta_5 - \frac{261}{320}\zeta_8\right)$$

Survey of primitive periods up to 11 loops

The Galois coaction on ϕ^4 periods (w. Oliver Schnetz)

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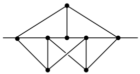
$\mathcal{P}\Big($  $\Big)$

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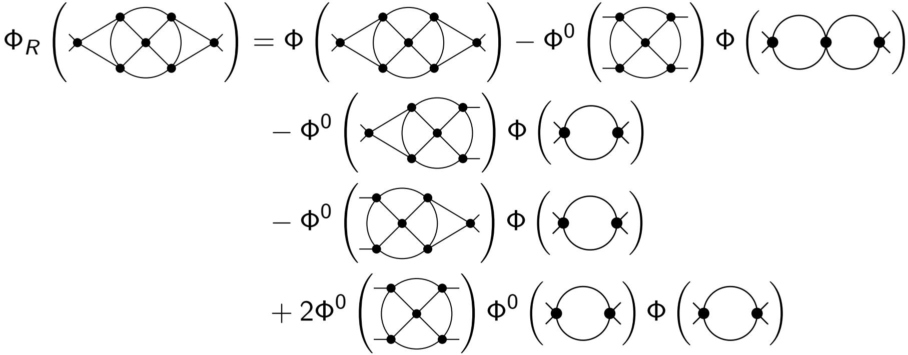
Survey of primitive periods up to 11 loops

The Galois coaction on ϕ^4 periods (w. Oliver Schnetz)

How to deal with divergences?

- Sector decomposition $\Leftarrow$ very tough & only numeric (at 6 loops)
- IBP to finite master integrals $\Leftarrow$ no IBP at 6 loops
- primitive linear combinations $\Leftarrow$ non-trivial to automate
- **one-scale BPHZ**

Renormalization of subdivergences:



BPHZ-like scheme

$\Phi^0(G) := \Phi(G)$ at a fixed renormalization point $(\vec{p_1}^{\,0}, \cdots, \vec{p_4}^{\,0}, m_0)$

Theorem (Renormalization under the integral sign, Weinberg '60)

The BPHZ-subtracted integrand is integrable. (This is false in MS!)

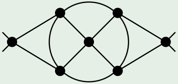
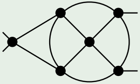
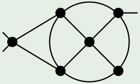
one-scale renormalization scheme

BPHZ renormalization of log. UV subdivergences via forest formula:

$$\Phi_R(G) = \sum_{F \in \mathcal{F}(G)} (-1)^F \prod_{\gamma \in G} \Phi^0(\gamma)\Phi(G/\gamma)$$

Idea [Brown & Kreimer '13]: Choose $\Phi^0(\gamma) := \Phi(\gamma^0)\big|_{p^2=1}$ to be 1-scale!

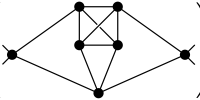
Example

$G =$  $\supset$ $\gamma =$  $\mapsto$ $\gamma^0 =$ 

- $\Phi_R(G)$ is a convergent integral at $\varepsilon = 0$
  $\Rightarrow$ `HyperInt` (ε-expansion under the integral sign)
- $\Phi(G) = \Phi_R(G) + \sum$ products of lower-loop p-integrals
- easy to implement

Automatization

1. compute forest formula & choose IR-safe one-scale structures γ^0
2. integrate the (convergent) $\partial_{p^2}\Phi_R(G)$ ($\Rightarrow$ `HyperInt`)
3. solve for $\Phi(G)$, using products of lower-loop integrals

$\mathcal{K}\mathcal{R}'\Bigg($  $\Bigg)$

$$= -\frac{\zeta_3}{3\varepsilon^4} + \left(\frac{5}{3}\zeta_3 + \frac{\pi^4}{180}\right)\frac{1}{\varepsilon^3} - \left(\frac{9}{2}\zeta_3 + \frac{7\pi^4}{360} - \frac{23}{6}\zeta_5\right)\frac{1}{\varepsilon^2} + \left(\frac{9}{2}\zeta_3 + \frac{7\pi^4}{360} - \frac{161}{30}\zeta_5 + \frac{7}{10}\zeta_3^2 - \frac{2\pi^6}{945}\right)\frac{1}{\varepsilon}$$

results

[Batkovich, Kompaniets, Chetyrkin '16]

$$\begin{aligned}\gamma_\phi^{\mathrm{MS}}(g) = {} & \frac{n+2}{36}g^2 - \frac{(n+8)(n+2)}{432}g^3 - \frac{5(n^2-18n-100)(n+2)}{5\,184}g^4 \\ & - \Big[1\,152(5n+22)\zeta_4 - 48(n^3-6n^2+64n+184)\zeta_3 \\ & \quad + (39n^3+296n^2+22\,752n+77\,056)\Big]\frac{(n+2)g^5}{186\,624} \\ & - \Big[512(2n^2+55n+186)\zeta_3^2 - 6\,400(2n^2+55n+186)\zeta_6 \\ & \quad + 4\,736(n+8)(5n+22)\zeta_5 \\ & \quad - 48(n^4+2n^3+328n^2+4\,496n+12\,912)\zeta_4 \\ & \quad + 16(n^4-936n^2-4\,368n-18\,592)\zeta_3 \\ & \quad + (29n^4+794n^3-30\,184n^2-549\,104n-1\,410\,544)\Big]\frac{(n+2)g^6}{746\,496} \\ & + \mathcal{O}\left(g^7\right)\end{aligned}$$

Check: large n-expansions [Vasilev, Pismak & Honkonen '81]
for β: [Broadhurst, Gracey & Kreimer '97]

Result ($N=1$), $D=4-2\varepsilon$

$$
\begin{aligned}
\beta^{\overline{\mathrm{MS}}}(g) = {} & -2\varepsilon g + 3g^2 - \tfrac{17}{3}g^3 + \left(\tfrac{145}{8} + 12\zeta_3\right) g^4 \\
& - \left(120\zeta_5 - 18\zeta_4 + 78\zeta_3 + \tfrac{3499}{48}\right) g^5 \\
& + \left(1323\zeta_7 + 45\zeta_3^2 - \tfrac{675}{2}\zeta_6 + 987\zeta_5 - \tfrac{1189}{8}\zeta_4 + \tfrac{7965}{16}\zeta_3 + \tfrac{764621}{2304}\right) g^6 \\
& - \left(\tfrac{46112}{3}\zeta_9 + 768\zeta_3^3 + \tfrac{51984}{25}\zeta_{3,5} - \tfrac{264543}{25}\zeta_8 + 4704\zeta_3\zeta_5 \right. \\
& \quad + \tfrac{63627}{5}\zeta_7 - 162\zeta_3\zeta_4 + \tfrac{8678}{5}\zeta_3^2 - \tfrac{6691}{2}\zeta_6 + \tfrac{63723}{10}\zeta_5 \\
& \quad \left. - \tfrac{16989}{16}\zeta_4 + \tfrac{779603}{240}\zeta_3 + \tfrac{18841427}{11520}\right) g^7 \\
& + \mathcal{O}\left(g^8\right)
\end{aligned}
$$

Numerical values: $\zeta_{3,5} = \sum_{1\le n<m} \frac{1}{n^3 m^5} \approx 0.037707673$

$$\approx -2\epsilon g + 3g^2 - 5.7g^3 + 32.6g^4 - 271.6g^5 + 2849g^6 - 34776g^7 + \mathcal{O}\left(g^8\right)$$

asymptotics

Let $\beta^{\mathrm{MS}}(g) = \sum_k \beta_k^{\mathrm{MS}}(-g)^k$.

Asymptotics of the perturbation series

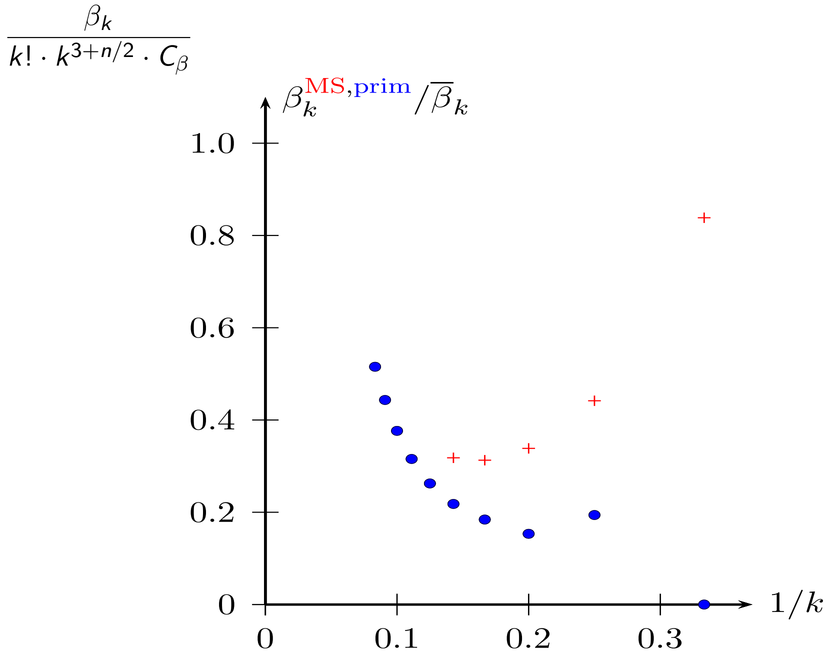
According to [McKane, Wallace & Bonfim '84],

$$\beta_k^{\mathrm{MS}} \sim \overline{\beta}_k := k! \cdot k^{3+n/2} \cdot C_\beta \quad \text{as } k \to \infty$$

where C_β is a constant that only depends on n:

$$C_\beta = \frac{36 \cdot 3^{(n+1)/2}}{\pi \Gamma(2+n/2) A^{2n+4}} \exp\left[-\frac{3}{2} - \frac{n+8}{3}\left(\gamma_{\mathrm{E}} + \frac{3}{4}\right)\right].$$

$\gamma_{\mathrm{E}} = \approx 0.577$ (Euler-Mascheroni) and $A \approx 1.282$ (Glaisher-Kinkelin)

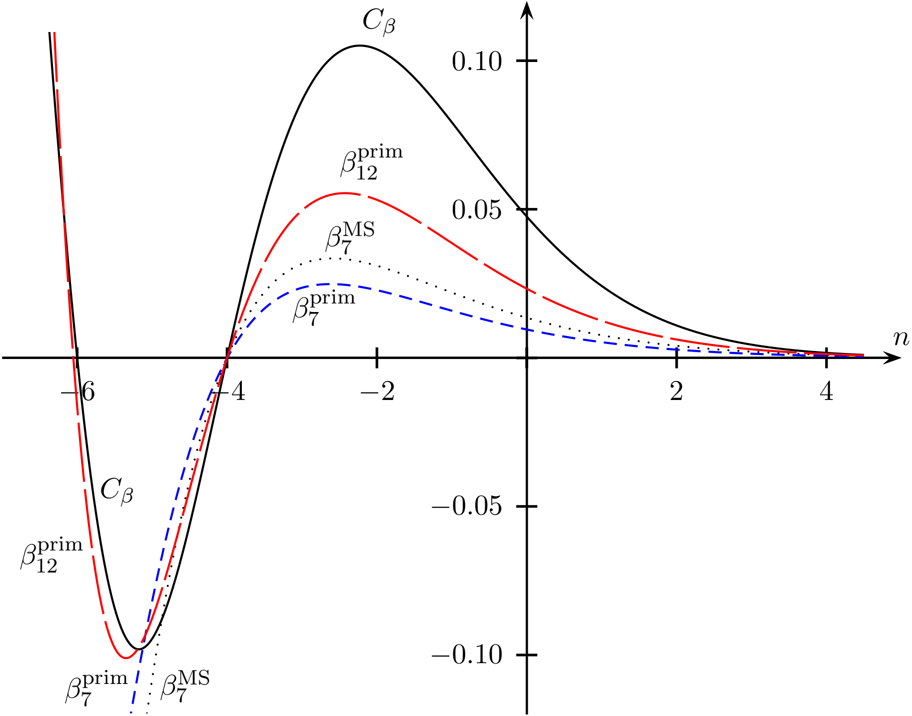


$\frac{\beta_k}{k! \cdot k^{3+n/2} \cdot C_\beta}$
$\beta_k^{\mathrm{MS},\mathrm{prim}} / \overline{\beta}_k$
1.0
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
$1/k$

loop order ℓ	1	2	3	4	5	6
$\beta^{\mathrm{MS}}_{\ell+1}/\overline{\beta}_{\ell+1}$ in %	548	83.5	43.8	33.5	30.9	31.4
$\beta^{\mathrm{MS}}_{\ell+1}$	3	5.67	32.5	272	2849	34776
$\beta^{\mathrm{prim}}_{\ell+1}$	3	0	14.4	124	1698	24130
$\beta^{\mathrm{prim}}_{\ell+1}/\beta^{\mathrm{MS}}_{\ell+1}$ in %	100	0	44.3	45.8	59.6	69.4
4-point graphs	1	2	8	26	124	627
primitives	1	0	1	1	3	10

$$C_\beta = \frac{36 \cdot 3^{(n+1)/2}}{\pi\Gamma(2+n/2)A^{2n+4}} \exp\left[-\frac{3}{2} - \frac{n+8}{3}\left(\gamma_{\mathrm{E}} + \frac{3}{4}\right)\right]$$

	loop order ℓ	first zero	second zero	third zero
$\beta^{\mathrm{MS}}_{\ell+1}(n)$	1	-8		
	2	-4.67		
	3	-4.025	-41.4	
	4	-4.020	-12.1	3219
	5	-4.0017	-8.76	-44.0
	6	-4.00044	-7.52	-20.0
$\beta^{\mathrm{prim}}_{\ell+1}(n)$	6	-3.99754	-7.22	-35.6
	7	-3.99982	-6.58	-15.1
	8	-3.99994	-6.31	-10.8
	9	-3.999997	-6.18	-9.24
	10	-3.99999991	-6.10	-8.55
	11	-4.000000095	-6.05	-8.21



C_β
0.10
$\beta_{12}^{\mathrm{prim}}$
0.05
β_7^{MS}
β_7^{prim}
n
-6
-4
-2
2
4
C_β
-0.05
$\beta_{12}^{\mathrm{prim}}$
-0.10
β_7^{prim}
β_7^{MS}

resummation

The ε-expansion $f(\varepsilon) = \sum_{n\geq 0} f_n \varepsilon^n$ of crit. exponents is divergent:

$$f_n \sim C n! a^n n^{b_0} \quad \text{[McKane, Wallace \& Bonfim '84]}$$

Borel-resummation after [Le Guillou & J. Zinn-Justin '85]:

$$f(\varepsilon) = \int_0^\infty x^{b-1} \tilde{f}(x) e^{-x/\varepsilon} \mathrm{d}x \quad \text{with} \quad \tilde{f}_n = \frac{f_n}{\Gamma(n+b)}$$

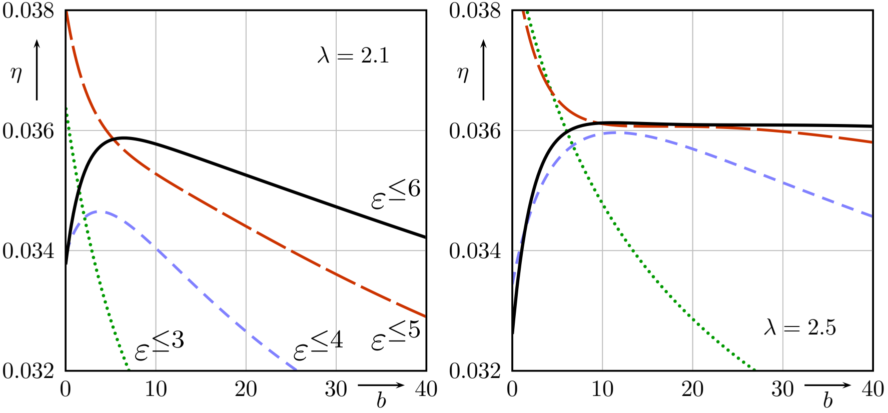
Conformal mapping (analytic continuation):

$$\tilde{f}(x) = \left(\frac{x}{w}\right)^\lambda \left(a_0 + a_1 w + \ldots + a_\ell w^\ell\right) \quad \text{where} \quad w(x) = \frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}$$

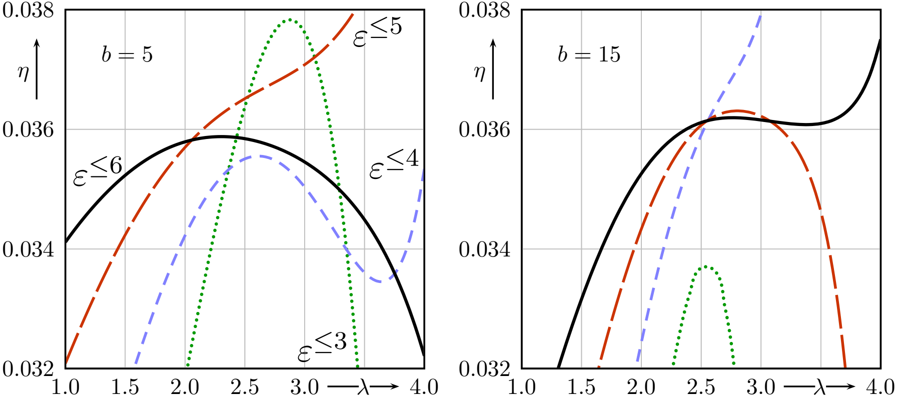
Homographic transformation: Re-expand in ε' given by

$$\varepsilon = \frac{\varepsilon'}{1+q\varepsilon'}$$

$$\eta(n=1) \approx 0.07407\varepsilon^2 + 0.1495\varepsilon^3 - 0.1333\varepsilon^4 + 0.8210\varepsilon^5 - 5.2015\varepsilon^6 + \mathcal{O}\left(\varepsilon^7\right)$$



$$\eta(n=1) \approx 0.07407\varepsilon^2 + 0.1495\varepsilon^3 - 0.1333\varepsilon^4 + 0.8210\varepsilon^5 - 5.2015\varepsilon^6 + \mathcal{O}\left(\varepsilon^7\right)$$



3d critical exponents

		$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
η		0.031043(3)	0.036298(2)	0.0381(2)	0.0378(3)	0.0360(3)
	ε^6	0.0310(7)	0.0362(6)	0.0380(6)	0.0378(5)	0.0366(4)
	ε^5	0.0314(11)	0.0366(11)	0.0384(10)	0.0382(10)	0.0370(9)
	G/ZJ	0.0300(50)	0.0360(50)	0.0380(50)	0.0375(45)	0.036(4)
ν		0.5875970(4)	0.629971(4)	0.6717(1)	0.7112(5)	0.7477(8)
	ε^6	0.5874(3)	0.6292(5)	0.6690(10)	0.7059(20)	0.7397(35)
	ε^5	0.5873(13)	0.6290(20)	0.6687(13)	0.7056(16)	0.7389(24)
	G/ZJ	0.5875(25)	0.6290(25)	0.6680(35)	0.7045(55)	0.737(8)
ω		0.904(5)	0.830(2)	0.811(10)	0.791(22)	0.817(30)
	ε^6	0.841(13)	0.820(7)	0.804(3)	0.795(7)	0.794(9)
	ε^5	0.835(11)	0.818(8)	0.803(6)	0.797(7)	0.795(6)
	G/ZJ	0.828(23)	0.814(18)	0.802(18)	0.794(18)	0.795(30)

2d critical exponents

		$n=-1$	$n=0$	$n=1$
η	N	0.15	0.208333...	0.25
	ε^6	0.130(17)	0.201(25)	0.237(27)
	ε^5	0.137(23)	0.215(35)	0.249(38)
	LeG/ZJ		0.21(5)	0.26(5)
ν	N	0.625	0.75	1
	ε^6	0.6036(23)	0.741(4)	0.952(14)
	ε^5	0.6025(27)	0.747(20)	0.944(48)
	LeG/ZJ		0.76(3)	0.99(4)
ω			2	1.75
	ε^6	1.95(28)	1.90(25)	1.71(9)
	ε^5	1.88(30)	1.83(25)	1.66(11)
	LeG/ZJ		1.7(2)	1.6(2)

Thanks

Thank you for your attention!

- new tools for massless propagators
- ϕ^4 beta function at six loops
- higher accuracy for critical exponents in $D = 3$

Thanks

Thank you for your attention!

- new tools for massless propagators
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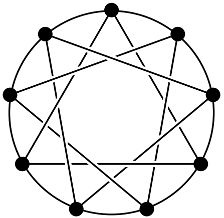
Stay tuned

tomorrow: ϕ^4 at 7 loops, by Oliver Schnetz

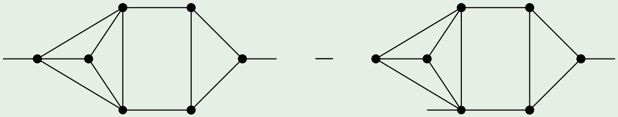


Alternative method

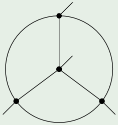
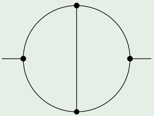
Given a graph G, find a linear combination X of graphs such that

1. $G - X$ is primitive (free of subdivergences) ($\Rightarrow$ HyperInt)
2. each term in X factorizes (has a ≥ 1 loop sub-p-integral) [Panzer '13]

Example



Both have the same subdivergence γ and quotient G/γ:

$\gamma =$  $G/\gamma =$ 

- simple: just p-integrals, no renormalization
- not straightforward to automate