Towards four-loop Standard Model renormalization in the gaugeless limit

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Motivation: Towards extremely high energies in the SM

- The SM being renormalizable can, in principle, be used to make predictions at scales $Q^2 \gg M_Z$, not accessible to colliders
- At such scales it is convenient to use running couplings $a(Q)$, which are obtained from a set of measured quantities $\{O\}$ by means of

1. Matching

- Relate running couplings $a(\mu \simeq M_Z)$ and $\{O\}$
- Initial values for evolution

2. Renormgroup

- Evolution of all SM couplings
- In $\overline{\mathrm{MS}}$ scheme, β-functions are simple and polynomial in $a(\mu)$

PDG 20XX		Fixed μ_0		
$\{O\} = M_b, M_W, M_Z,$ M_H, M_t, G_F	$\longrightarrow$	$g_i(\mu_0), y_i(\mu_0), \lambda(\mu_0)$ in $\overline{MS}$ scheme	$\longrightarrow$	Evolve from μ_0 to scale μ

Motivation: SM vacuum stability analysis at NNLO

- Three loop beta-functions for gauge Yukawa and self-coupling

  [Mihaila,Salomon,Steinhauser'12;Bednyakov,AFP,Velizhanin'12,13;Chetyrkin,Zoller'13]

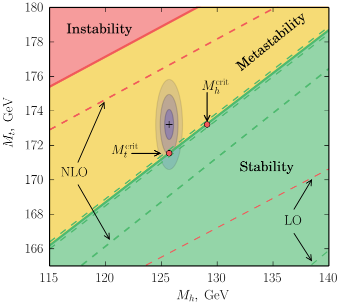
- Two loop full $\mathcal{O}(\alpha^2)$ threshold corrections

  [Buttazzo et al.'13;Kniehl,AFP,Veretin'15]

$$V_{\text{eff}}(\phi \gg v) \simeq \frac{\lambda(\mu = \phi)}{4}\phi^4$$

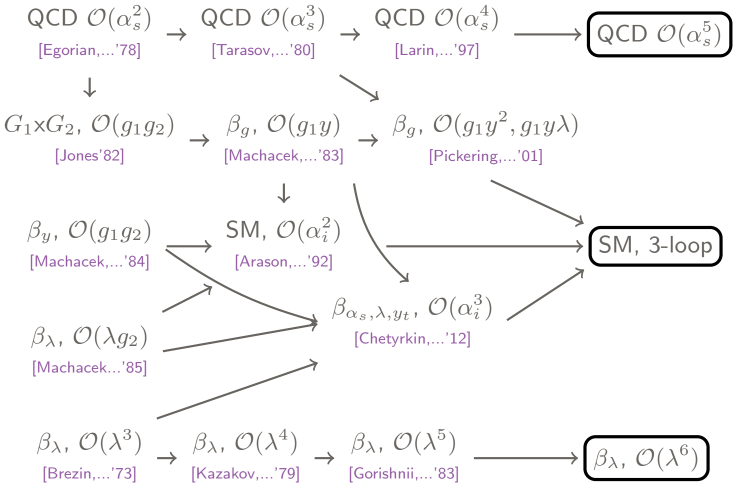
$$(4\pi)^2 \frac{d\lambda}{d\ln\mu^2} = 12\lambda + 6y_t^2\lambda - 3y_t^4 + \ldots$$

$$(4\pi)^2 \frac{dy_t}{d\ln\mu^2} = \frac{9}{4}y_t^3 - 4{g_s}^2 y_t + + \ldots$$



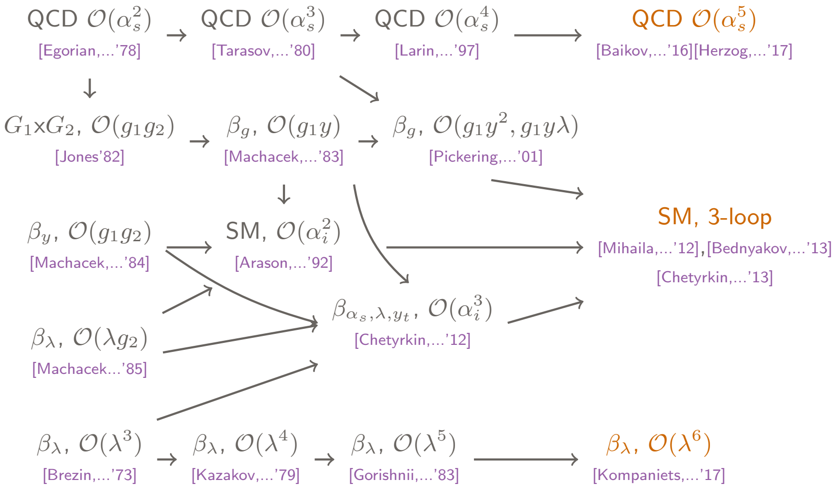
Main steps of beta-functions evaluation

Few years ago:



Main steps of beta-functions evaluation

Today:



Next step: calculation of four-loop SM beta-functions

What is known at four-loops?

- Strong coupling beta-function in the SM gaugeless limit — This talk

  [Bednyakov,AFP'15][Zoller'15]

- Leading QCD corrections to β_λ

  [Martin'15][Chetyrkin,Zoller'16]

- Generalization of QCD results on reducible fermion representaions

  [Chetyrkin,Zoller'17]

- Pure QCD corrections to β_y and Higgs field anomalous dimension

  [Larin,. . . '97][Chetyrkin'97][Chetyrkin'96]

Are these corrections dominating?

Strong coupling — matching and running: known results

- Running in $\overline{\mathrm{MS}}$
  - 1-loop [Gross,Wilczek'73, Politzer'73]
  - 2-loop [Jones'74,Egorian,...'78] (QCD), [Machachek,Vaughn'83] (SM)
  - 3-loop [Tarasov,...'80] (QCD), [Mihaila,...'12,Bednyakov,...'12] (SM)
  - 4-loop [van Ritbergen,Vermaseren,Larin'97,Czakon'04] (QCD)
  - 5-loop [Baikov,...'16,Luthe,...'16,Herzog,...'17] (QCD)
- Matching in $\overline{\mathrm{MS}}$: we "match" effective five-flavor ($n_f = 5$) QCD and a more fundamental theory (usually, QCD with top quark $n_f = 6$)

$$\alpha_s^{(5)}(\mu) = \alpha_s(\mu)\xi_{\alpha_s}(\mu, M)$$

"integrate out" heavy fields with mass M

  - 1/2-loop [Bernreuther,Wetzel'81-83], [Bednyakov'15] (2-loop SM)
  - 3-loop [Chetyrkin,Kniehl,Steinhauser'97-98]
  - 4-loop [Schroder,...'05,Chetyrkin,...'05,Kniehl,...'06]

NB: L-loop running + (L-1)-loop matching are self-consistent

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Four-loop strong coupling beta-function: from QCD to SM

- **Starting point:** Limit of vanishing $SU(2)\times U(1)$ gauge couplings
  Only the following SM parameters are considered

$$a_i = \left(\frac{g_s^2}{16\pi^2}, \frac{y_t^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G\right)$$

  Significantly reduce number of diagrams!
  NB: we keep track of gauge-parameter dependence!

- Easier to track γ_5 in dimensional regularization (see below):
  - no γ_5 in gauge vertices!
  - γ_5 appears only in (pseudo/charged) scalar couplings.

- In what follows (h counts powers of a_i)

$$\frac{d\ a_s}{d\ \log\mu^2} = \beta_{a_s} = -\sum_{i=0}^{3}\beta_i h^{i+2}$$

Further simplifications: background field gauge

- Split gauge fields $V = \tilde{V} + \hat{V}$ in
  - quantum $\tilde{V} = (\tilde{G}, \tilde{W}, \tilde{B}, \dots)$ and
  - background $\hat{V} = (\hat{G}, \hat{W}, \hat{B}, \dots)$
- Background fields do not propagate
- Modified Feynman rules [Abbot'80]
- QED-like connection between renormalization constants

$$Z_{a_s} = 1/Z_{\hat{G}}, \qquad Z_{\xi_G} = Z_{\tilde{G}}$$

- Only two-point functions are required (given all 3-loop Z-factors)
- Multiplicative renormalization by $a_{\text{bare}} = Z_a a_{\text{ren}}$

$$\Gamma^{(l)}_{\text{ren}} = Z^{(l)}_{\Gamma} \left[1 + \Gamma^{(1)}_{\text{bare}}(a_{\text{bare}}) + \Gamma^{(2)}_{\text{bare}}(a_{\text{bare}}) + \cdots + \Gamma^{(l)}_{\text{bare}}(a_{\text{bare}})\right]$$

- No need to apply IRR trick, have access to finite parts

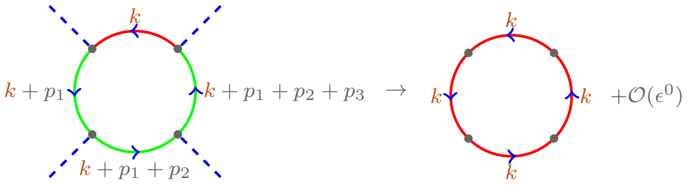
RGE from propagator-type integrals

- Three-loop experience (different approaches to IRR [Vladimirov'78])
  - Massless propagators: gauge couplings, field renormalization constants
    using `FORM` based `MINCER` package
    NB: finite parts are available (in massless/unbroken theory)
  - Three-loop massive bubbles: Yukawa and Higgs self-coupling
    using `FORM` based `MATAD` package
    NB: aux mass renormalization
- Four-loop experience
  - QCD beta function
    massive vacuum integrals available, possible to calculate all types of renormalization constants.
  - Massless propagators. Difficult to prepare reduction. Easy to formulate the problem
    Independent tool for two-point Green functions renormalization constants calculation

Setup v.1

- Model file of the SM in BFG tested at lower loop calculations
- `DIANA/QGRAF` [Nogueira'93;Fleischer,Tentyukov'99]
  Diagram generation
- Prepared set of mappings to 3 auxiliary topos
  each topo with 11 denominators and 3 irreducible numerators
- Reduction
  - `LiteRed`
    IBP rules preparation [Lee'12]
  - `FIRE5, C++` version
    Integral reduction [Smirnov'14]
  - Master integrals
    4-loop propagators [Baikov,Chetyrkin'10;Lee,Smirnovs'11]

Four-loop RGE from fully massive tadpoles



$$\underbrace{\frac{1}{(k+p)^2 - M^2}}_{\omega=-2} = \underbrace{\frac{1}{k^2 - m_A^2}}_{\omega=-2} + \underbrace{\frac{M^2 - p^2 - 2kp - m_A^2}{k^2 - m_A^2} \frac{1}{(k+p)^2 - M^2}}_{\omega=-3}$$

- Four-loop QCD beta-function [Ritbergen,Vermaseren,Larin'97][Czakon'04]
- Anom. dim. of twist-2 opertors in QCD and N=4 SYM [Velizhanin'14]
- Renormalization of QCD with extended fermion sector [Chetyrkin,Zoller'17]

Setup v.2: improved reduction

We have four independent ways for reduction:

1. Reduction of massless propagators
   - LiteRed
     IBP rules preparation [Lee'12]
   - FIRE5, C++ version
     Integral reduction [Smirnov'14]
   - Master integrals
     4-loop propagators [Baikov,Chetyrkin'10;Lee,Smirnovs'11]
2. Reduction of massles propagators with FORM package FORCER
   Parametric integral reduction [Ruijl,Ueda,Vermaseren'17]
3. Reduction of fully massive tadpoles
   - LiteRed
     IBP rules preparation [Lee'12]
   - FIRE5, C++ version
     Integral reduction [Smirnov'14]
   - Master integrals
     4-loop fully massive tadpoles [Czakon'04]
4. Reduction of four-loop tadpoles with new FORM code FMFT
   details below [AFP'17]

4 loop QCD β-function and renormalization constants

- IRR with auxiliary mass
  fully massive four-loop tadpoles, possible to calculate all renormalization constants
  - From Z_g, Z_c, Z_{ccg} [Larin,Vermaseren,Ritbergen'97]

$$Z_{a_g} = \frac{Z_{ccg}^2}{Z_c^2 Z_g}$$

  - From Z_g, Z_q, Z_{qqg} [Czakon'04]

$$Z_{a_g} = \frac{Z_{qqg}^2}{Z_q^2 Z_g}$$

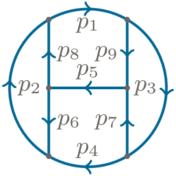
- Using 3-loop massless integrals [Chetyrkin'04]
  - From Z_c, Z_{ccg} and already known β_{a_g}
    Impossible to calculate Z_g, but independent calculation of other RCs

$$Z_g = \frac{Z_{ccg}^2}{Z_c^2 Z_{a_g}}$$

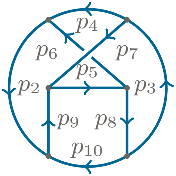
- Using 4-loop massless propagator type integrals and BFG

$$Z_{a_s} = 1/Z_{\hat{G}}, \qquad Z_{\xi_G} = Z_{\tilde{G}}$$

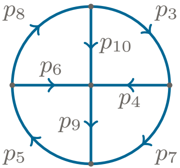
- Topolgies **H**,**X**,**BMW** need manual reduction rules



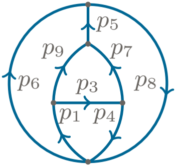
H



X



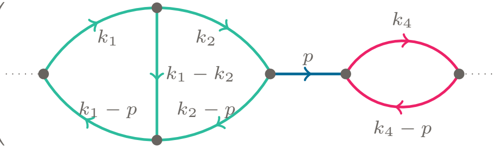
BMW



FG

- Topolgy **FG** and all its subtopologies can be expressed as convolution

$$J_{\mathrm{FG}} = \int d[p] \left(\right.$$



$$\left. \right)$$

- One-loop and two-loop propagator type integrals with massive lines can be reduced separately

FMFT: topology FG and performance

- Using dimensional shifts we can reduce numerators of both one- and two-loop subintegrals [Tarasov,97]
- Remaining integrals with arbitrary power of denominator with momenta p reduced using one dimensional recurence relations

Available for download:

`http://git.io/fmft`

Nonplanar integral $X(-n, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with numerator

$n =$	3	4	5	6	7	8
FMFT	0:00:11	0:00:27	0:01:55	0:07:35	0:25:31	01:30:31
FIRE	0:01:58	0:09:10	0:28:17	2:16:42	9:19:57	46:42:29

Time format **hh:mm:ss**, FIRE used with LiteRed rules

Renormalization constants in gaugeless limit of SM

- γ_{H} - Higgs field anomalous dimension from Z_{H}
- β_{λ} - Higgs self-coupling beta-function from Z_{λ}

$$Z_{\lambda} = \frac{Z_{\mathrm{HHHH}}}{Z_{\mathrm{H}}^2}$$

- β_{y_t} - top Yukawa coupling beta-function from Z_{y_t}

$$Z_{y_t} = \frac{Z_{\mathrm{uuH}}}{\sqrt{Z_{\mathrm{u,L}} Z_{\mathrm{u,R}} Z_{\mathrm{H}}}}$$

- β_{m^2} - Higgs mass parameter beta-function from Z_{m^2}

$$Z_{m^2} = \frac{Z_{\mathrm{HH[HH]}}}{Z_{\mathrm{H}}}$$

- Green functions calculated using massless propagators:

$$\Gamma_{\mathrm{HH}}, \Gamma_{\mathrm{uu}}, \Gamma_{\mathrm{uuH}}$$

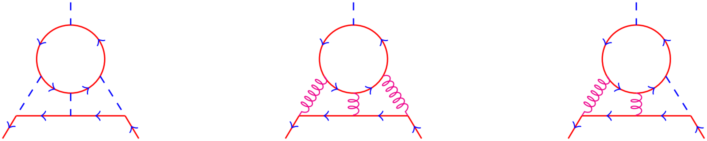
- Green functions calculated using massive tadpoles:

$$\Gamma_{\mathrm{HH}}, \Gamma_{\mathrm{uu}}, \Gamma_{\mathrm{uuH}}, \Gamma_{\mathrm{HHHH}}, \Gamma_{\mathrm{HH[HH]}}$$

Traces at three-loop level

For γ_5 in dimensional regularization we use naive treatment, except diagrams with two fermionic traces with four contracted indices

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5]\,\mathrm{tr}[\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5]$$



$\sim \mathrm{tr}[\gamma^\alpha\gamma^\beta\gamma^5]$ | $2n+1$ γ matrices | $\sim \epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}$

- Substitution is correct for $D = 4$

$$\mathrm{tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5] = -4i\epsilon_{\mu\nu\rho\sigma}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -\mathcal{T}^{[\mu\nu\rho\sigma]}_{[\alpha\beta\gamma\delta]}, \qquad \mathcal{T}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} = \delta^\mu_\alpha\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\delta\,,$$

- Apply in $D = 4 - 2\epsilon$, but for diagrams w/o subdivergencies

Number of diagrams up to four loops

- Total number of diagrams and additional diagrams with $\varepsilon \otimes \varepsilon$ contraction

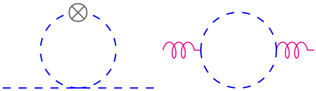
	1	2	$3/\gamma_5$	$4/\gamma_5$
$\Gamma_{\hat{g}\hat{g}}$	4	43	867	25374/72
Γ_{uu}	4	39	920	30035/94
Γ_{HH}	1	14	276	8822/18
Γ_{uuH}	3	102	4030/18	185981/2048
Γ_{HHHH}	5	47	1307	46536/74
$\Gamma_{\mathrm{HH[HH]}}$	3	27	616	23044/18

- Number of diagrams for Γ_{HHHH} can be reduced due to graph symmetries using `GraphState` package. Original numbers:

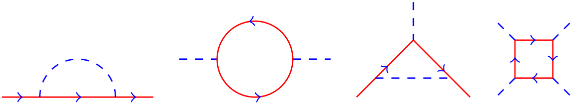
	1	2	3	4
Γ_{HHHH}	15	327	13212	685599

Uncertainty in contributions from lower loops

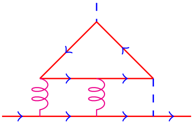
- No contribution from one-loop diagrams



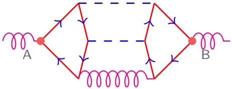
- Determinent contribution from one-loop renormalization (Z_{y_t} insertion)



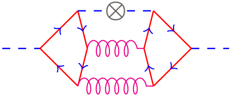
- Undetermined contribution from three-loop renormalization (Z_g, Z_{y_t} insertion)



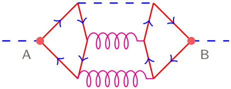
Four-loop diagrams and reading points



(**A** and **B**) or (**A** and not **B**) or (not **A** and not **B**)



fixed



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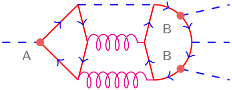

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For quark propagator and `uuH` vertex situation is more complicated

Keeping trace of uncertainties

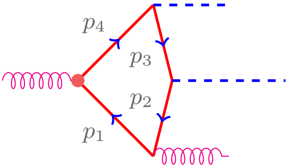
Two generalizations of trace taking procedures:

1. Use tensor reduction and move trace outside the integral

$$\int d[p_{1\ldots n}]\, \mathrm{tr}[\not{p}_1 \ldots \not{p}_k] \to \mathrm{tr}[\gamma_\alpha \ldots \gamma_\beta] \int d[p_{1\ldots n}](p_i \cdot p_j)$$

   - Safe to apply $d = 4$ rules if integral have no higher poles
   - Different reading points corresponds to different integrals combinations

2. Cut fermion line in all possible ways, but take trace under integral sign



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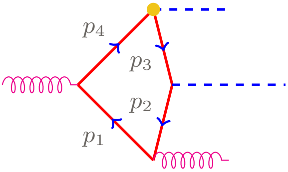

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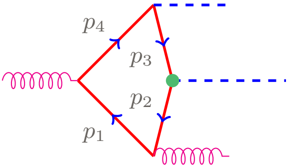

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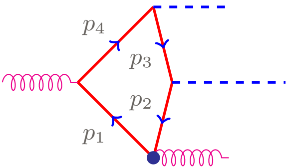
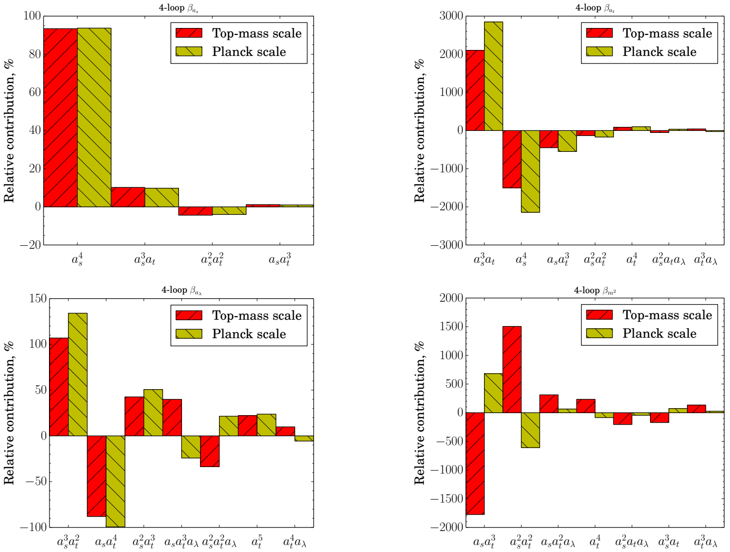


Relative contributions



Relative loop contributions

Substituting running couplings at scale $\mu = M_t$ and separating contribution from diagrams with two fermion traces requiring non naive treatment:

- Contribution from part with uncertainty due to γ_5 in β_{a_s} negligible
- Large cancellations, no uncertainty

$$\frac{\beta_{m^2}}{\beta_0} = 1.h \quad + \quad 0.0264h^2 \quad + \quad 0.00266h^3 \quad - \quad [0.000027 - 0.000066]h^4$$

- Small cancellation and small contribution from piece with uncertainty

$$\frac{\beta_{a_\lambda}}{\beta_0} = 1.h \quad + \quad 0.0089h^2 \quad - \quad 0.00025h^3 \quad - \quad [0.001323 \pm 0.000048]h^4$$

- Large cancelations and large contribution from piece with uncertainty

$$\frac{\beta_{a_t}}{\beta_0} = 1.h \quad + \quad 0.1529h^2 \quad + \quad 0.00743h^3 \quad + \quad [0.000019 \pm 0.000612]h^4$$

Conclusions

- We calculated universal contribution not affected by γ_5 definition in dimensional regularization to four-loop SM beta-functions in gaugeless limit
- Part dependent on γ_5 definition paramtrized for different reading points and need further analysis
- Four-loop package `FMFT` for fully massive tadpoles reduction created and successfully applied for calcualtion of all needed Green functions