Evaluating multiloop Feynman integrals by differential equations

Vladimir A. Smirnov

Skobeltsyn Institute of Nuclear Physics of Moscow State University

Paris, June 8, 2017

Based on collaboration with Burkhard Eden, Roman Lee, Alexander Smirnov, Matthias Steinhauser, João Caetano, Ömer Gürdoğan, Vladimir Kazakov

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- Conclusion

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A lot of applications [J.M. Henn, A.V. Smirnov, V.A. Smirnov, K. Melnikov, F. Caola, R. Bonciani, V. Del Duca, H. Frellesvig, F. Moriello, M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, T. Gehrmann, A. von Manteuffel, E. Weihs, F. Dulat, B. Mistlberger, R. N. Lee,...]

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices) $\rightarrow$
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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Public codes to solve IBP relations

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Private codes to solve IBP relations

- Gehrmann & Remiddi, Laporta, Czakon, Schröder, Pak, Sturm, Marquard & Seidel, Velizhanin, Mistlberger,. . . , von Manteuffel

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After this: solve DE.

Let $f = (f_1, \ldots, f_N)$ be *primary* master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

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$$\partial_i f(\epsilon, x) = \epsilon\, A_i(x) f(\epsilon, x)\,.$$

In the differential form,

$$d\, f(\epsilon, x) = \epsilon\, (d\, \tilde{A}(x))\, f(x, \epsilon)\,,$$

where

$$\tilde{A} = \sum_k \tilde{A}_k\, \log(\alpha_k)\,.$$

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Let us call it *epsilon form* and the corresponding basis *canonical*.

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where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

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For example, if $x_k = 0, -1, 1$ the matrix $\tilde{A}$ involves
$\log(x), \log(x \pm 1)$
and results for elements of such a basis are expressed in terms of harmonic polylogarithms (HPL) by E. Remiddi and J. Vermaseren.

Solve DE order in order in ε

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Solution

$$f^{(i)}(x) = \int_0^x A(x') f^{(i-1)}(x') \mathrm{d}x' + c_i$$

$$\int_{0\le\tau_1\le\ldots\tau_k\le x} \mathrm{d}\tilde{A}(\tau_k)\ldots\cdot\ldots\mathrm{d}\tilde{A}(\tau_1)$$

→ a linear combination of integrals

$$\int_{0\le\tau_1\le\ldots\tau_k\le x} \frac{\mathrm{d}\tau_k}{\tau_k+a_k}\ldots\frac{\mathrm{d}\tau_1}{\tau_1+a_1}$$

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HPLs

$$H(a_1, a_2, \ldots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \ldots, a_n; t)\mathrm{d}t\,,$$

where $f(\pm 1; t) = 1/(1 \mp t), \;\; f(0; t) = 1/t,$

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  Adjust a transformation matrix using a proper Ansatz.

Evaluating QCD form factors.
Three-loop results
[P. A. Baikov, K. G. Chetyrkin, A.V. Smirnov, V.S. & M. Steinhauser'09,
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Analytic results for the three-loop master integrals up to weight 8
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motivated by a future four-loop calculation.

The photon-quark form factor in the large-N_c limit.
[J. Henn, A. Smirnov, V.S. & M. Steinhauser'16;
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non-planar calculations in $\mathcal{N} = 4$ SYM with numerical methods
[R. Boels, Gang Yang & T. Huber'17]

Photon quark and Higgs quark form factors

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}\left(\not{q}_2\,\Gamma_q^{\mu}\,\not{q}_1\,\gamma_{\mu}\right)\,,$$

$$F_b(q^2) = -\frac{1}{2q^2}\mathrm{Tr}\left(\not{q}_2\,\Gamma_b\,\not{q}_1\,\right)\,.$$

$q = q_1 + q_2$; q_1 and q_2 are the incoming quark and antiquark momenta and q is the momentum of the photon.

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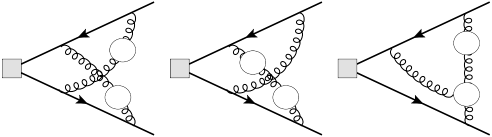
n_f^2-contributions



$$
\begin{aligned}
\log(F_x) = \\
&\frac{\alpha_s}{4\pi}\left\{\frac{1}{\epsilon^2}\left[-\frac{1}{2}C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\gamma^0_x\right]\right\}\\
&+\left(\frac{\alpha_s}{4\pi}\right)^2\left\{\frac{1}{\epsilon^3}\left[\frac{3}{8}\beta_0 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^2}\left[-\frac{1}{2}\beta_0\gamma^0_x-\frac{1}{8}C_F\gamma^1_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^1_x}{2}\right]\right\}\\
&+\left(\frac{\alpha_s}{4\pi}\right)^3\left\{\frac{1}{\epsilon^4}\left[-\frac{11}{36}\beta_0^2 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^3}\left[C_F\left(\frac{2}{9}\beta_1\gamma^0_{\text{cusp}}+\frac{5}{36}\beta_0\gamma^1_{\text{cusp}}\right)+\frac{1}{3}\beta_0^2\gamma^0_x\right]\right.\\
&\left.+\frac{1}{\epsilon^2}\left[-\frac{1}{3}\beta_1\gamma^0_x-\frac{1}{3}\beta_0\gamma^1_x-\frac{1}{18}C_F\gamma^2_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^2_x}{3}\right]\right\}\\
&+\left(\frac{\alpha_s}{4\pi}\right)^4\left\{\frac{1}{\epsilon^5}\left[\frac{25}{96}\beta_0^3 C_F\gamma^0_{\text{cusp}}\right]+\frac{1}{\epsilon^4}\left[C_F\left(-\frac{13}{96}\beta_0^2\gamma^1_{\text{cusp}}-\frac{5}{12}\beta_1\beta_0\gamma^0_{\text{cusp}}\right)-\frac{1}{4}\beta_0^3\gamma^0_x\right]\right.\\
&+\frac{1}{\epsilon^3}\left[C_F\left(\frac{5}{32}\beta_2\gamma^0_{\text{cusp}}+\frac{3}{32}\beta_1\gamma^1_{\text{cusp}}+\frac{7}{96}\beta_0\gamma^2_{\text{cusp}}\right)+\frac{1}{4}\beta_0^2\gamma^1_x+\frac{1}{2}\beta_1\beta_0\gamma^0_x\right]\\
&\left.+\frac{1}{\epsilon^2}\left[-\frac{1}{4}\beta_2\gamma^0_x-\frac{1}{4}\beta_1\gamma^1_x-\frac{1}{4}\beta_0\gamma^2_x-\frac{1}{32}C_F\gamma^3_{\text{cusp}}\right]+\frac{1}{\epsilon}\left[\frac{\gamma^3_x}{4}\right]\right\}+\ldots,
\end{aligned}
$$

where $x \in \{q, b\}$ and $\mu^2 = -q^2$

The cusp and collinear anomalous dimensions

$$\gamma_x = \sum_{n\geq 0} \left(\frac{\alpha_s(\mu^2)}{4\pi} \right)^n \gamma_x^n \,,$$

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The coefficients of the β function are

$$\begin{aligned}
\beta_0 &= \frac{11C_A}{3} - \frac{2n_f}{3}\,,\\
\beta_1 &= -\frac{10C_A n_f}{3} + \frac{34C_A^2}{3} - 2C_F n_f\,,\\
\beta_2 &= -\frac{205}{18}C_A C_F n_f - \frac{1415}{54}C_A^2 n_f + \frac{79}{54}C_A n_f^2 + \frac{2857C_A^3}{54} + \frac{11}{9}C_F n_f^2 + C_F^2 n_f\,.
\end{aligned}$$

Our results

$$
\begin{aligned}
\gamma^{0}_{\mathrm{cusp}} &= 4\,, \\
\gamma^{1}_{\mathrm{cusp}} &= \left(\frac{268}{9}-\frac{4\pi^2}{3}\right)C_A-\frac{40n_f}{9}\,, \\
\gamma^{2}_{\mathrm{cusp}} &= C_A^2\left(\frac{88\zeta_3}{3}+\frac{44\pi^4}{45}-\frac{536\pi^2}{27}+\frac{490}{3}\right) \\
&\quad + n_f\left[C_A\left(-\frac{112\zeta_3}{3}+\frac{80\pi^2}{27}-\frac{836}{27}\right)+C_F\left(32\zeta_3-\frac{110}{3}\right)\right]-\frac{16n_f^2}{27}\,, \\
\gamma^{3}_{\mathrm{cusp}} &= \gamma^{3,n_f^0}_{\mathbf{cusp}}+\gamma^{3,n_f^1}_{\mathbf{cusp}}n_f \\
&\quad + n_f^2\left[C_A\left(\frac{2240\zeta_3}{27}-\frac{56\pi^4}{135}-\frac{304\pi^2}{243}+\frac{923}{81}\right)+C_F\left(-\frac{640\zeta_3}{9}+\frac{16\pi^4}{45}+\frac{2392}{81}\right)\right] \\
&\quad + \left(\frac{64\zeta_3}{27}-\frac{32}{81}\right)n_f^3
\end{aligned}
$$

$$
\begin{aligned}
\gamma_q^0 &= -3C_F\,, \\
\gamma_q^1 &= \left(26\zeta_3 - \frac{11\pi^2}{6} - \frac{961}{54}\right) C_A C_F + \left(\frac{65}{27} + \frac{\pi^2}{3}\right) C_F n_f + \left(-24\zeta_3 + 2\pi^2 - \frac{3}{2}\right) C_F^2\,, \\
\gamma_q^2 &= n_f \left[\left(-\frac{964\zeta_3}{27} + \frac{11\pi^4}{45} + \frac{1297\pi^2}{243} - \frac{8659}{729}\right) C_A C_F + \left(\frac{256\zeta_3}{9} - \frac{14\pi^4}{27} - \frac{13\pi^2}{9}\right.\right. \\
&\quad \left.\left. + \frac{2953}{54}\right) C_F^2 \right] + \left(-\frac{8}{3}\pi^2\zeta_3 - \frac{844\zeta_3}{3} - 120\zeta_5 + \frac{247\pi^4}{135} + \frac{205\pi^2}{9} - \frac{151}{4}\right) C_A C_F^2 \\
&\quad + \left(-\frac{44}{9}\pi^2\zeta_3 + \frac{3526\zeta_3}{9} - 136\zeta_5 - \frac{83\pi^4}{90} - \frac{7163\pi^2}{486} - \frac{139345}{2916}\right) C_A^2 C_F \\
&\quad + \left(-\frac{8\zeta_3}{27} - \frac{10\pi^2}{27} + \frac{2417}{729}\right) C_F n_f^2 + \left(\frac{16\pi^2\zeta_3}{3} - 68\zeta_3 + 240\zeta_5 - \frac{8\pi^4}{5} - 3\pi^2\right. \\
&\quad \left. - \frac{29}{2}\right) C_F^3\,, \\
\gamma_q^3 &= n_f^2 \left[\left(-\frac{64}{27}\pi^2\zeta_3 - \frac{7436\zeta_3}{243} + \frac{592\zeta_5}{9} - \frac{19\pi^4}{135} - \frac{41579\pi^2}{8748} + \frac{97189}{34992}\right) C_A C_F \right. \\
&\quad \left. + \left(\frac{56\pi^2\zeta_3}{27} + \frac{2116\zeta_3}{81} - \frac{520\zeta_5}{9} + \frac{1004\pi^4}{1215} - \frac{493\pi^2}{81} - \frac{9965}{972}\right) C_F^2 \right] \\
&\quad + \left(-\frac{712\zeta_3}{243} - \frac{16\pi^4}{1215} - \frac{4\pi^2}{81} + \frac{18691}{6561}\right) C_F n_f^3 + n_f \gamma_q^{3,n_f^1} + \gamma_q^{3,n_f^0}
\end{aligned}
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The coefficients $\gamma_{\rm cusp}^{3,n_f^0}$, $\gamma_{\rm cusp}^{3,n_f^1}$, γ_q^{3,n_f^0} and γ_q^{3,n_f^1} are only known in the large-N_c limit
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Agreement with the known n_f^3 terms of γ_{cusp}^3
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Calculations in generic ξ-gauge for checks.

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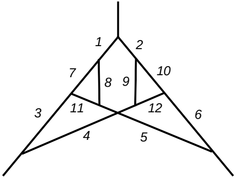

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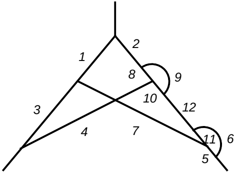

786

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- We write down solutions of these differential equations in a straightforward way order-by-order in ϵ in terms of HPL with letters 0 and 1.

- We fix the boundary conditions for the canonical master integrals at the point $x = 1$ where integrals are analytic and reduce to well-known propagator integrals
[P.A. Baikov & K.G. Chetyrkin'10;
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- From the analytic results for the naive part we obtain analytical results for the sought-after one-scale master integrals after changing back to the primary basis.

Examples of our results for the master integrals

$$G^{(786)}_{111011011110} = + \frac{1}{\epsilon}\left[\frac{7\pi^4\zeta_3}{360} - \frac{5\pi^2\zeta_5}{3} - \frac{441\zeta_7}{16}\right] + \left[-\frac{87 s_{8a}}{2} - \frac{23}{6}\pi^2\zeta_3^2 - \frac{473\zeta_5\zeta_3}{2} - \frac{8069\pi^8}{777600}\right],$$

$$G^{(786)}_{111011011120} = + \frac{1}{\epsilon^5}\left[-\frac{\pi^2}{96}\right] + \frac{1}{\epsilon^4}\left[\frac{7\zeta_3}{16}\right] + \frac{1}{\epsilon^3}\left[\frac{463\pi^4}{8640}\right] + \frac{1}{\epsilon^2}\left[\frac{1247\zeta_5}{48} - \frac{127\pi^2\zeta_3}{144}\right]$$
$$+ \frac{1}{\epsilon}\left[\frac{38761\pi^6}{362880} - \frac{1079\zeta_3^2}{12}\right] - \frac{78923\pi^4\zeta_3}{12960} + \frac{18301\pi^2\zeta_5}{720} - \frac{161\zeta_7}{24}$$
$$+ \epsilon\left[\frac{3291 s_{8a}}{5} + \frac{55183}{216}\pi^2\zeta_3^2 - \frac{330689\zeta_5\zeta_3}{90} - \frac{122374187\pi^8}{653184000}\right].$$

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Correlation functions $\to$ both scattering amplitudes and the dual polygonal Wilson loops [L. F. Alday, B. Eden, G. P. Korchemsky, J. Maldacena & E. Sokatchev'11, B. Eden, G. P. Korchemsky & E. Sokatchev]

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The complexity increases very much at higher loops.
An explicit result for the two-loop four-point stress-tensor correlator
[B. Eden, C. Schubert & E. Sokatchev'00, M. Bianchi, S. Kovacs, G. Rossi & Y. S. Stanev'00]

Three-loop calculations [B. Eden, P. Heslop, G. P. Korchemsky & E. Sokatchev'12, J. Drummond, C. Duhr, B. Eden, P. Heslop, J. Pennington & V. A. Smirnov'13, D. Chicherin, J. Drummond, P. Heslop & E. Sokatchev'15]

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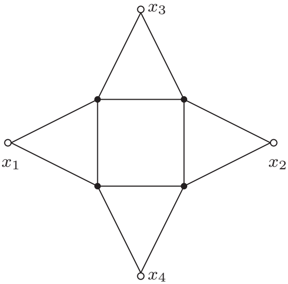
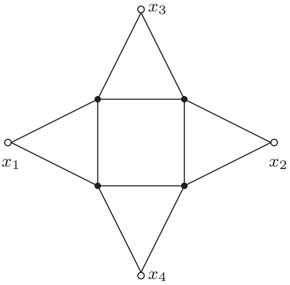
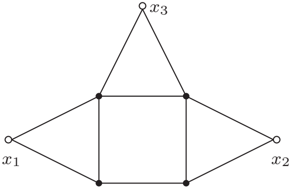
Four loops: 26 genuine four-loop integrals in the planar part of the correlator five of which can be related to the ladder with four rungs by flip identities on subintegrals.


x_3
x_1
x_2
x_4



$x_4 \to \infty$; $x_3 = 0$



Evaluating by DE using D-dimensional IBP relations [B. Eden & V.S.'16].

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The goals:

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- To evaluate this four-dimensional four-loop integral.
- To evaluate the whole set of the master integrals in D dimensions. (The first example of such a calculation.)

$$F_{a_1,\ldots,a_{18}} = \int \ldots \int \frac{\mathrm{d}^D x_5\, \mathrm{d}^D x_6\, \mathrm{d}^D x_7\, \mathrm{d}^D x_8}{[-x_5^2]^{a_1}[-x_6^2]^{a_2}[-(x_1-x_5)^2]^{a_3}[-(x_1-x_7)^2]^{a_4}}$$
$$\times \frac{[-(x_2-x_5)^2]^{-a_{11}}[-(x_1-x_6)^2]^{-a_{12}}[-(x_2-x_7)^2]^{-a_{13}}[-(x_6-x_7)^2]^{-a_{14}}}{[-(x_2-x_6)^2]^{a_5}[-(x_2-x_8)^2]^{a_6}[-(x_5-x_6)^2]^{a_7}[-(x_5-x_7)^2]^{a_8}}$$
$$\times \frac{[-x_7^2]^{-a_{15}}[-(x_1-x_8)^2]^{-a_{16}}[-(x_5-x_8)^2]^{-a_{17}}[-x_8^2]^{-a_{18}}}{[-(x_6-x_8)^2]^{a_9}[-(x_7-x_8)^2]^{a_{10}}} \; .$$

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F_{a_1,\ldots,a_{18}} \; = \; & \int\ldots\int \frac{\mathrm{d}^D x_5\, \mathrm{d}^D x_6\, \mathrm{d}^D x_7\, \mathrm{d}^D x_8}{[-x_5^2]^{a_1}[-x_6^2]^{a_2}[-(x_1-x_5)^2]^{a_3}[-(x_1-x_7)^2]^{a_4}} \\
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Three coordinate differences squared off the light cone
$x_1^2 = -z\bar{z}, \quad x_2^2 = -(1-z)(1-\bar{z}), \quad (x_1-x_2)^2 = -1$

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Our integral is $F_{1,1,1,1,1,1,1,1,1,1,0,\ldots}$

FIRE $\to$ 213 master integrals

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Constructing a canonical basis

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We used a code constructed by Burkhard Eden.

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Constructing a canonical basis
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212 of 213 elements of a canonical basis were obtained with this code.

DE in our canonical basis

$$\frac{\partial}{\partial z_1} f = \varepsilon \bar{A}_1(z_1, z_2) f \,,$$
$$\frac{\partial}{\partial z_2} f = \varepsilon \bar{A}_2(z_1, z_2) f \,.$$

DE in our canonical basis

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$\bar{A}_i = \frac{\partial}{\partial z_i} \tilde{A}$
with

$$\tilde{A} = \sum_k \tilde{A}_k \, \log(\alpha_k) \,.$$

and letters taken from the alphabet
$\{z_1, 1-z_1, z_2, 1-z_2, -z_1+z_2, 1-z_1-z_2, 1-z_1 z_2, z_1+z_2-z_1 z_2\}$

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First, solve

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Solution

$$f^{(i)}(z_1, z_2) = \int_0^{z_1} d\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-1)}(\bar{z}_1, z_2) + h^{(i)}(z_2)$$

The result is a linear combination of multiple (Goncharov) polylogarithms (MPL) $G(a_1, a_2, \ldots, a_w; z_1)$
where $a_i \in \{0, 1, z_2, 1 - z_2, 1/z_2, -z_2/(1 - z_2)\}$.

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$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

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$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n; t)$$

Substitute solution into the second equation to obtain

$$\begin{aligned} \frac{\partial}{\partial z_2} h^{(i)}(z_2) = \bar{A}_2(z_1, z_2) h^{(i-1)}(z_2) \\ + \bar{A}_2(z_1, z_2) \int_0^{z_1} \mathrm{d}\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-2)}(\bar{z}_1, z_2) \\ - \frac{\partial}{\partial z_2} \int_0^{z_1} \mathrm{d}\bar{z}_1 \bar{A}_1(\bar{z}_1, z_2) f^{(i-1)}(\bar{z}_1, z_2) \end{aligned}$$

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If y is the expansion parameter in the limit $z_1, z_2 \to 0$ then we encounter the following power dependence
$y^0, y^{-\varepsilon}, y^{-2\varepsilon}, y^{-3\varepsilon}, y^{-4\varepsilon}$

An interplay between expansion by regions
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and solving canonical DE in the given limit.

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One could consider also the short-distance limit $x_2 \to 0$ and the limits $z_1 \to 0, z_2 \to 1$ and $z_1 \to 1, z_2 \to 0$ which are light-cone limits $x_1^2 \to 0$ and $x_2^2 \to 0$.

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It turns out that the information about the the Euclidean limit $x_1 \to 0$ is sufficient.

The result for our integral is $(z-\bar{z})^{-2}$ times a linear combination of single valued multiple polylogarithms [F. C. S. Brown'04]

$$\mathcal{L}_{\{a_1,\ldots,a_8\}} = (-1)^{\sum a_i} G(a_1,\ldots a_8; z) + \sum c_{ij}\, G(\underline{a}_i; z)\, G(\underline{a}_j; \bar{z})$$

where $\underline{a}_i \cup \underline{a}_j$ has length 8 and $\underline{a}_j$ is never the empty word.

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The coefficients c_{ij} are polynomials of multiple zeta values such that all branch cuts cancel. The entries in the weight vectors are in the set $\{0,1\}$ and the "condensed notation" $\ldots 0,0,0,1\ldots = \ldots 4 \ldots$ etc. is used

After flipping points $x_2 \leftrightarrow x_3$ (i.e. $z \to 1/z$, $\bar{z} \to 1/\bar{z}$) followed by $x_1 \leftrightarrow x_2$ (which implies $z \to 1-z$, $\bar{z} \to 1-\bar{z}$): this function takes the form

$$\begin{aligned} - & \mathcal{L}_{\{3,5\}} + \mathcal{L}_{\{5,3\}} + \mathcal{L}_{\{2,5,0\}} - \mathcal{L}_{\{4,3,0\}} - \mathcal{L}_{\{1,5,0,0\}} + \mathcal{L}_{\{3,3,0,0\}} \\ - & \mathcal{L}_{\{2,3,0,0,0\}} + \mathcal{L}_{\{1,3,0,0,0,0\}} \end{aligned}$$

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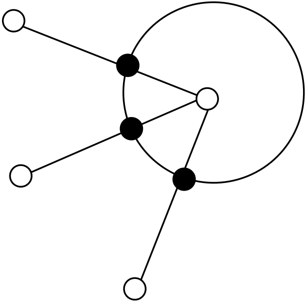
Agreement with independent calculations by O. Schnetz and E. Panzer.

The sputnik diagram



The evaluation of a 3-loop coordinate-space integral contributing to the conformal 4-point correlation function in the so-called bi-scalar CFT, an integrable theory in 4 dimensions obtained in a special limit of twisted $N = 4$ SYM.
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The integral is

$$\int \ldots \int \frac{\mathrm{d}^D y_1 \, \mathrm{d}^D y_2 \, \mathrm{d}^D y_3}{(x_1 - y_1)^2 (x_2 - y_2)^2 (x_3 - y_3)^2 (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2}$$

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Its pole part is $2\zeta(3)/\varepsilon$.

The integral is $F_{1,1,1,1,1,1,0,0,0,0,0,0}$, where

$$F_{a_1,\ldots,a_{12}} = \int\ldots\int \frac{\mathrm{d}^D y_1\, \mathrm{d}^D y_2\, \mathrm{d}^D y_3}{[-(x_1-y_1)^2]^{a_1}[-(x_2-y_2)^2]^{a_2}[-y_3^2]^{a_3}[-(y_1-y_2)^2]^{a_4}}$$
$$\times \frac{1}{[-(y_2-y_3)^2]^{a_5}[-(y_1-y_3)^2]^{a_6}[-(x_1-y_2)^2]^{a_7}[-(x_1-y_3)^2]^{a_8}}$$
$$\times \frac{1}{[-(x_2-y_1)^2]^{a_9}[-(x_2-y_3)^2]^{a_{10}}[-y_1^2]^{a_{11}}[-y_2^2]^{a_{12}}}$$

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with $x_1^2 = -z\bar{z}, \quad x_2^2 = -(1-z)(1-\bar{z}), \quad (x_1-x_2)^2 = -1.$
The same procedure as in the previous four-loop calculation.

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Constructing a canonical basis with the Eden's code

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Constructing a canonical basis with the Eden's code
DE in our canonical basis

$$
\begin{aligned}
\frac{\partial}{\partial z_1} f &= \varepsilon \bar{A}_1(z_1, z_2) f \,, \\
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\end{aligned}
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$\bar{A}_i = \frac{\partial}{\partial z_i} \tilde{A}$
with

$$\tilde{A} = \sum_k \tilde{A}_k \log(\alpha_k) \,.$$

and letters taken from the alphabet
$\{1 - z_1, 1 - z_2, z_1 - z_2\}$

The result for the finite part of our integral is

```
(1/(30 (z1 - z2)))(\[Pi]^4 z1 - \[Pi]^4 z2 -
  30 z2 G[0, 1, z2] G[1, 0, z1] + 30 z1 z2 G[0, 1, z2] G[1, 0, z1] +
  30 z1 G[0, 1, z1] G[1, 0, z2] - 30 z1 z2 G[0, 1, z1] G[1, 0, z2] +
  30 G[1, 0, z2] G[1, 1, z1] - 30 z1 G[1, 0, z2] G[1, 1, z1] -
  30 z2 G[1, 0, z2] G[1, 1, z1] + 30 z1 z2 G[1, 0, z2] G[1, 1, z1] +
  30 z2 G[0, 0, z2] (z1 G[0, 1, z1] - (-1 + z1) G[1, 1, z1]) -
  30 G[1, 0, z1] G[1, 1, z2] + 30 z1 G[1, 0, z1] G[1, 1, z2] +
  30 z2 G[1, 0, z1] G[1, 1, z2] - 30 z1 z2 G[1, 0, z1] G[1, 1, z2] -
  30 z1 G[0, 0, z1] (z2 G[0, 1, z2] - (-1 + z2) G[1, 1, z2]) -
  30 z1 z2 G[0, z2] G[0, 0, 1, z1] - 30 z1 G[1, z2] G[0, 0, 1, z1] +  30 z1 z2 G[1, z2] G[0, 0, 1, z1] +
  30 z1 z2 G[0, z1] G[0, 0, 1, z2] + 30 z2 G[1, z1] G[0, 0, 1, z2] - 30 z1 z2 G[1, z1] G[0, 0, 1, z2] +
  30 z1 z2 G[0, z2] G[0, 1, 0, z1] + 30 z1 G[1, z2] G[0, 1, 0, z1] -  30 z1 z2 G[1, z2] G[0, 1, 0, z1] -
  30 z1 z2 G[0, z1] G[0, 1, 0, z2] - 30 z2 G[1, z1] G[0, 1, 0, z2] +
  30 z1 z2 G[1, z1] G[0, 1, 0, z2] - 30 z2 G[0, z2] G[1, 0, 1, z1] +
  30 z1 z2 G[0, z2] G[1, 0, 1, z1] - 30 G[1, z2] G[1, 0, 1, z1] +
  30 z1 G[1, z2] G[1, 0, 1, z1] + 30 z2 G[1, z2] G[1, 0, 1, z1] -
  30 z1 z2 G[1, z2] G[1, 0, 1, z1] + 30 z1 G[0, z1] G[1, 0, 1, z2] -
  30 z1 z2 G[0, z1] G[1, 0, 1, z2] + 30 G[1, z1] G[1, 0, 1, z2] -
  30 z1 G[1, z1] G[1, 0, 1, z2] - 30 z2 G[1, z1] G[1, 0, 1, z2] +
  30 z1 z2 G[1, z1] G[1, 0, 1, z2] + 30 z2 G[0, z2] G[1, 1, 0, z1] -
  30 z1 z2 G[0, z2] G[1, 1, 0, z1] + 30 G[1, z2] G[1, 1, 0, z1] -
  30 z1 G[1, z2] G[1, 1, 0, z1] - 30 z2 G[1, z2] G[1, 1, 0, z1] +
  30 z1 z2 G[1, z2] G[1, 1, 0, z1] - 30 z1 G[0, z1] G[1, 1, 0, z2] +
  30 z1 z2 G[0, z1] G[1, 1, 0, z2] - 30 G[1, z1] G[1, 1, 0, z2] +
  30 z1 G[1, z1] G[1, 1, 0, z2] + 30 z2 G[1, z1] G[1, 1, 0, z2] -
  30 z1 z2 G[1, z1] G[1, 1, 0, z2] - 30 z1 z2 G[0, 0, 1, 0, z1] +
  30 z1 z2 G[0, 0, 1, 0, z2] - 30 z1 G[0, 0, 1, 1, z1] + 30 z1 z2 G[0, 0, 1, 1, z1] + 30 z2 G[0, 0, 1, 1, z2] -
  30 z1 z2 G[0, 0, 1, 1, z2] + 30 z1 z2 G[0, 1, 0, 0, z1] - 30 z1 z2 G[0, 1, 0, 0, z2] + 30 z1 G[0, 1, 0, 1, z1] -
  30 z1 z2 G[0, 1, 0, 1, z1] - 30 z2 G[0, 1, 0, 1, z2] + 30 z1 z2 G[0, 1, 0, 1, z2] - 30 z2 G[1, 0, 1, 0, z1] +
  30 z1 z2 G[1, 0, 1, 0, z1] + 30 z1 G[1, 0, 1, 0, z2] - 30 z1 z2 G[1, 0, 1, 0, z2] - 30 G[1, 0, 1, 1, z1] +
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  30 z1 z2 G[1, 1, 0, 0, z1] - 30 z1 G[1, 1, 0, 0, z2] + 30 z1 z2 G[1, 1, 0, 0, z2] + 30 G[1, 1, 0, 1, z1] -
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  30 z1 G[1, 1, 0, 1, z2] + 30 z2 G[1, 1, 0, 1, z2] - 30 z1 z2 G[1, 1, 0, 1, z2] + 540 z1 Zeta[3] - 540 z2 Zeta[3] +
  180 G[1, z1] Zeta[3] - 180 z1 G[1, z1] Zeta[3] - 180 G[1, z2] Zeta[3] + 180 z2 G[1, z2] Zeta[3])
```

The static potential of two heavy quarks

$$V^{[c]}(|\vec{q}\,|) = -4\pi C^{[c]} \frac{\alpha_s(|\vec{q}\,|)}{\vec{q}^{\,2}} \left[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1^{[c]} + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi} \right)^2 a_2^{[c]} + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi} \right)^3 \left(a_3^{[c]} + 8\pi^2 C_A^3 \ln \frac{\mu^2}{\vec{q}^{\,2}} \right) + \cdots \right],$$

with $C^{[1]} = C_F$ for the colour-singlet and $C^{[8]} = C_F - C_A/2$ for the colour-octet case. Here, $C_A = N_c$ and $C_F = (N_c^2 - 1)/(2N_c)$ are the eigenvalues of the quadratic Casimir operators of the adjoint and fundamental representations of the $SU(N_c)$ colour gauge group, respectively.

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$a_3^{[c]} = a_3^{[c],(3)}n_l^3 + a_3^{[c],(2)}n_l^2 + a_3^{[c],(1)}n_l + a_3^{[c],(0)}$
where n_l is the number of light (massless) quarks.

a_3 with three constants evaluated numerically
[A. Smirnov, V.S. & M. Steinhauser'10, C. Anzai, Y. Kiyo & Y. Sumino'10]

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In particular, for $N_c = 3$ we have

$$\begin{aligned} a_3^{[1],(0)} \quad = \quad & \frac{385645}{108} + \pi^2 \left[\frac{893}{3} + 816\tilde{a}_4 + l_2 \left(1844 - 1302\zeta(3)\right) + 295\zeta(3) \right] \\ & + 5256\zeta(3) + \pi^4 \left(-\frac{227}{20} + 115 l_2 + 35 l_2^2 \right) - \frac{17343\zeta(5)}{2} \\ & - \frac{1643\pi^6}{168} - \frac{3861(\zeta(3))^2}{2} + 3888 s_6 \,, \end{aligned}$$

with $s_6 = \zeta(-5,-1) + \zeta(6)$, $\tilde{a}_4 = \mathrm{Li}_4(1/2) + \frac{l_2^4}{24}$, $l_2 = \log 2$

Feynman integrals depending on two vectors, q and v, but the scales q^2 and v^2 are separated because $v \cdot q = 0$.

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$$F_{a_1,\ldots,a_{12}} = \iiint \frac{\mathrm{d}^D k \, \mathrm{d}^D l \, \mathrm{d}^D r}{(-k^2)^{a_1}(-l^2)^{a_2}(-r^2)^{a_3}(-(r+q)^2)^{a_4}(-(k-l+r+q)^2)^{a_5}}$$
$$\times \frac{(-v \cdot l)^{-a_{10}}(-(k-r)^2)^{-a_{12}}}{(-(k+q)^2)^{a_6}(-(l-r)^2)^{a_7}(-(k-l)^2)^{a_8}(y/2 - v \cdot k)^{a_9}(y/2 - v \cdot r)^{a_{11}}} \, .$$

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Derive and solve DE with respect to y and obtain $F_{1,\ldots,1,,1,-2,1,0}$
[R. Lee & V.S.'16]

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