Higher-loop momodromy relations

Pierre Vanhove

IPhT CEA-Saclay

Workshop on multiloop calculations, June 8, 2017

based on the works

[0907.1425], [1003.2403], [1010.3933], [1608.016653]

work done and work in progress with

N.E.J. Bjerrum-Bohr, P. Damgaard, T. Søndergaard, Piotr Tourkine, Alexandre Ochirov

I just put 1.795372 and 2.204628 together.
And what does that mean?
Four!
(Doctor Who)

It is crucial for experimental and theoretical reasons to have efficient methods for evaluating amplitudes of physical processes in quantum field theory

- multilegs and multiloop amplitudes for LHC physics
- Quantum gravity: simplification of amplitudes $\mathcal{N} = 8$ supergravity and pure gravity

Unfortunately the number of individual Feynman graphs rises dramatically with the number of external legs or loop order, and tensor reduction methods increase the number of terms even more.

A huge number of cancellation are needed to get the result leading to

- instabilities due to large numerical cancellations in matrix elements
- obfuscation of the fundamental structure of the interactions: gauge invariance, ultraviolet divergences, infrared singularities, hidden symmetries

Explicit amplitude computations display rather unexpectedly simple structures

- One-loop multi-photon in QED and multi-graviton amplitudes in $\mathcal{N} = 8$ supergravity amplitudes share the *same* no-triangle property [Bjerrum-Bohr, Vanhove],[Badger,Bjerrum-Bohr,Vanhove]
- Simpler than expected subleading color contribution at one-loop for Higgs + 2 jets, $\bar{q}qggH$, process [Badger, Campbell, Ellis, Williams]
- Better UV behaviour of subleading color factor amplitudes at multi-loop order in $\mathcal{N} = 4$ SYM [Berkovits, Green, Russo, Vanhove]
- Better UV behaviour of multiloop supergravity amplitudes [Bern et al.]

All these simplifications hints on simple structures than the diagrammatics from Feynman rules suggest

Amplitudes from dressed φ^3 cubic vertices

[Bern, Carrasco, Johansson] have proposed a parametrisation of gauge and gravity amplitude based

on n-points, L-loop skeleton graphs, $\gamma \in \Gamma_n^L$, with only cubic φ^3 vertices dressed by

- n_γ: lorentz factor build from external and loop momenta and polarisations
- c_γ: a color factor

- At tree level in gauge and gravity

$$\mathcal{A}_n^{\text{tree}} = \sum_{\gamma \in \Gamma_n^{\text{tree}}} \frac{n_\gamma c_\gamma}{\prod_{\alpha \in \gamma} p_\alpha^2}; \qquad \mathcal{M}_n^{\text{tree}} = \sum_{\gamma \in \Gamma_n^{\text{tree}}} \frac{n_\gamma \tilde{n}_\gamma}{\prod_{\alpha \in \gamma} p_\alpha^2}$$

Amplitudes from dressed φ^3 cubic vertices

[Bern, Carrasco, Johansson] have proposed a parametrisation of gauge and gravity amplitude based

on n-points, L-loop skeleton graphs, $\gamma \in \Gamma_n^L$, with only cubic φ^3 vertices dressed by

- n_γ: lorentz factor build from external and loop momenta and polarisations
- c_γ: a color factor

- at loop order for $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity

$$\mathcal{A}_n^L = \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma c_\gamma}{\prod_{\alpha\in\gamma} p_\alpha^2}; \qquad \mathcal{M}_n^L = \sum_{\gamma \in \Gamma_n^L} \int d^{3L-3}\ell_\alpha \frac{n_\gamma \tilde{n}_\gamma}{\prod_{\alpha\in\gamma} p_\alpha^2}$$

Colour/kinematic duality

The [BCJ] parametrisation is based on a colour kinematic duality between the colour factors c_γ and the kinematic factors n_γ
The kinematic factor satisfy a dual Jacobi identity

$$n_s + n_t + n_u = 0$$

Dual to the corresponding identity on the colour factor

$$c_s + c_t + c_u = 0$$

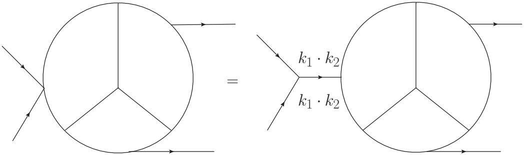
This duality allowed to construct four graviton amplitudes in maximal supergravity up to four loop orders.
The rule needed to be changed for the construction of the five loops integrand [Bern et al.]

Amplitudes from dressed φ^3 cubic vertices

Gauge invariance, supersymmetry, crossing symmetry of the amplitudes requires *contact terms* given by higher point vertices

Such contact terms are resolved by inserting appropriate factors

$$1 = \frac{k_1 \cdot k_2}{k_1 \cdot k_2} :$$



the same manipulations for internal vertices as well lead to graphs build on φ^3 skeletons with dressing numerator factors characteristic of the theory one considers.

Higher-loop colour/kinematic duality

- What are the systematics behind the colour/kinematic duality?
- What is the origin of the difficulties behind the construction of 5-loop amplitude in $\mathcal{N} = 8$ supergravity?
- What is a systematic construction at any loop orders?

Part I

Tree-level amplitudes

Tree-level amplitudes in gauge theory

Tree-level gauge theory amplitudes are decomposed as in color ordered factors

$$\mathcal{A}_n^{\text{tree}} = g_{\text{YM}}^{n-2} \sum_{\sigma \in \mathfrak{S}_n/\mathbb{Z}_n} \text{Tr}(\lambda^{\sigma(1)} \cdots \lambda^{\sigma(n)}) \, A_n(\sigma(1), \cdots, \sigma(n))$$

All the information are in the color ordered partial amplitude

$$A_n^{\sigma} := A_n(\sigma(1), \cdots, \sigma(n)); \qquad \sigma \in \mathfrak{S}_n$$

The color ordered amplitudes are not independent and satisfy relations kinematic relations

- Cyclicity property

$$A_n(1,\ldots,n) = A_n(2,n,\ldots,1)$$

Reduces the number of independent amplitudes to $(n-1)!$

- Reflection property

$$A_n(1,\ldots,n) = (-1)^n A_n(n,\ldots,1)$$

Reduces further to $\frac{1}{2}(n-1)!$ independent amplitudes

- Photon decoupling identity

$$\sum_{\sigma\in\mathfrak{S}_n} A_n(\sigma(1),\ldots,\sigma(n)) = 0$$

What is the minimal number of independent amplitude ?

Tree-level amplitudes in gauge theory

Instead of considering the sum of the mutilple field theory graph individualy we treat the field theory amplitudes and the infinite tension limit $\alpha' \to 0$ of the tree-level string amplitudes

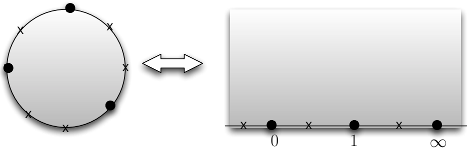
$$A_{\rm SYM}(\sigma(1),\dots,\sigma(n)) = \lim_{\alpha'\to 0} \mathfrak{A}(\sigma(1),\dots,\sigma(n))$$

$$\mathfrak{A}(\sigma(1),\dots,\sigma(n)) = \left\langle U^{(1)}(z_1)U^{(n-1)}(z_{n-1})U^{(n)}(z_n)\prod_{i=2}^{n-2}\int_{\rm ordered} d^2z_i V^{(i)}\right\rangle$$

where $U(z)$ and $V(z)$ are vertex operators and $\langle\cdots\rangle$ is the path integral over the world-sheet fields.

This can applied to any string theory formalism (Bosonic, RNS, Green-Schwarz, pure spinor, ...) in any spacetime dimensions

Open string tree-level amplitude on the disc



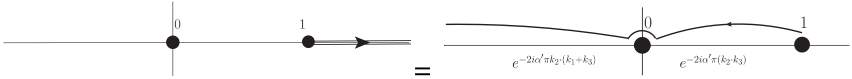
$PSL(2,\mathbb{R})$ invariance $z_1 = 0$, $z_{n-1} = 1$ and $z_n = +\infty$. (3 marked points)

$$\mathfrak{A}(1,\dots,n) = \int_{x_1<\dots<x_n} \prod_{i=2}^{n-2} dx_i \prod_{1\leqslant i<j\leqslant n} (x_i - x_j)^{2\alpha' k_i\cdot k_j} \sum_{(\zeta_j)\in\{0,1,x_i\}} L_k \prod_{i=2}^{n-2} \frac{1}{x_j - \zeta_j}$$

- The L_k factor encodes the information on the theory we consider: scalar or vector in the adjoint, etc

Monodromies from contour deformation

Contour deformation [Bjerrum-bohr, Damgaard, Vanhove; Stieberger]



- The real and imaginary part of the monodromy relations lead to a set of linear system of equations

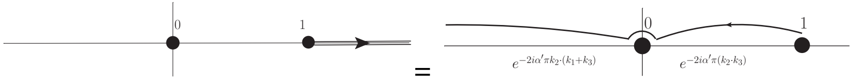
$$\mathfrak{A}_n(\beta_1,\ldots,\beta_r,1,\alpha_1,\ldots,\alpha_s,n) = (-1)^r \times$$
$$\times \Re\mathrm{e}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)} \mathfrak{A}_n(1,\{\sigma\},n)\Big]$$

$$0 = \Im\mathrm{m}\Big[\prod_{1\leqslant i<j\leqslant r} e^{(\beta_i\cdot\beta_j)} \sum_{\sigma\subset \mathrm{OP}\{\alpha\}\cup\{\beta^T\}} \prod_{i=1}^{r}\prod_{j=1}^{s} e^{(\alpha_i,\beta_j)} \mathfrak{A}_n(1,\{\sigma\},n)\Big]$$

$\exp(\alpha,\beta) = \exp(2i\pi\alpha' k_\alpha\cdot k_\beta)$ if $\Re\mathrm{e}(z_\beta - z_\alpha) > 0$ or 1 otherwise

Monodromies from contour deformation

Contour deformation [Bjerrum-bohr, Damgaard, Vanhove; Stieberger; Bjerrum-bohr, Damgaard, Søndergaard, Vanhove]



- This leads to an object name momentum kernel $\mathcal{S}$

$$\mathcal{S}_{\alpha'}[i_1,\dots,i_k|j_1,\dots,j_k]_p \equiv \prod_{t=1}^{k} \frac{1}{\alpha'} \sin \alpha'(p\cdot k_{i_t} + \sum_{q>t}^{k} \theta(i_t,i_q)\, k_{i_t}\cdot k_{i_q})$$

- This leads to the following set of constraints on the string theory amplitudes for all $\beta\in\mathfrak{S}_{n-2}$

$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}_{\alpha'}[\sigma(2,\dots,n-1)|\beta(2,\dots,n-1)]_{k_1} \mathcal{A}_n(n,\sigma(2,\dots,n-1),1) = 0$$

Minimal Basis for tree-level amplitudes

- This leads to a linear system of rank $(n-3)!$ in the amplitudes
- All ordered amplitudes can be expanded in the *minimal* basis B_n

$$(B_n)^{\sigma} := A(1, \underbrace{\sigma(2), \ldots, \sigma(n-2)}_{permutation}, n-1, n); \qquad \sigma \in \mathfrak{S}_{n-3}$$

$$A_n^{\sigma} = \sum_{\sigma' \in \mathfrak{S}_{n-3}} S^{\sigma}_{\sigma'} \, (B_n)^{\sigma'}$$

- These monodromy relations apply for *all* matter content

[Bjerrum-bohr, Damgaard, Vanhove; Stieberger; Bjerrum-bohr, Damgaard, Søndergaard, Vanhove; Stieberger, Schloterer; Mafra, Stieberger, Schloterer]

The gravity amplitudes

The [Kawai, Lewellen, Tye] factorization of the closed string amplitude in term of open string amplitude leads to the following expression for the closed string tree amplitudes

$$\mathcal{M}_n \sim \sum_{\sigma\in\mathfrak{S}_{n-3}} \sum_{\gamma\in\mathfrak{S}_j} \sum_{\beta\in\mathfrak{S}_{n-3-j}} \mathcal{S}_{\alpha'}[\gamma\circ\sigma|\sigma]_{k_1} \mathcal{S}_{\alpha'}[\beta\circ\sigma|\sigma]_{k_{n-1}}$$
$$\times\ \mathcal{A}_n(1,\sigma(\dots),n-1,n)\, \widetilde{\mathcal{A}}_n(\gamma\circ\sigma,1,n-1,\beta\circ\sigma,n)\,.$$

- The expression is independent of j thanks to the annihilation relation

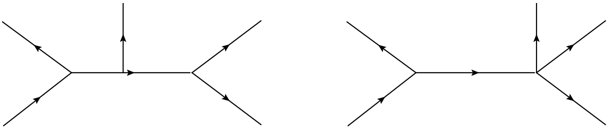
$$\sum_{\sigma\in\mathfrak{S}_{n-2}} \mathcal{S}_{\alpha'}(\sigma)\mathcal{A}(\sigma) = 0$$

- The expression is a sum over $(n-3)! \times (j-2)! \times (n-1-j)!$ terms.
- The number of terms takes the maximal value $(n-3)! \times (n-3)!$ for $j=2$ or $j=n-1$. This is the most symmetric case

The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1),\ldots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$



The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1), \ldots, \sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The numerator factors are *not* gauge invariant

- The monodromy relations between the color ordered amplitudes imply

$$0 = (s_{13} + s_{23}) A_5(1,2,3,4,5) - s_{35} A_5(1,2,4,3,5) + s_{13} A_5(1,3,2,4,5)$$

The five-point case

- We consider the for color ordered gauge amplitudes

$$A_5(\sigma(1),\ldots,\sigma(5)) = \sum_{i=1}^{5} \frac{n_{r_i}}{p_{1,i}^2 p_{2,i}^2}$$

- The system is solved by the generalized dual Jacobi relations

$$X_{ijk} = n_i - n_j + n_k = P_{n-4}(s_{ij}); \qquad c_i - c_j + c_k = 0$$

- The n_i are not uniquely defined by the pairing $n_i c_i$ summed over the graph gives the gauge invariant amplitudes
- $P_{n-4} = 0$ is the [Bern, Carrasco, Johansson] solution
- $P_{n-4} \neq 0$ is a degree $n-4$ polynomial from the freedom in resolving the higher point vertices [Tye, Zhang], [Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

The five-point case

- The double copy for gravity puts some constraints on the polynomials

$$\mathcal{M}_n^{\text{tree}} = \sum_{\gamma \in \Gamma_n^{\text{tree}}} \frac{n_\gamma \tilde{n}_\gamma}{\prod_{\alpha \in \gamma} p_\alpha^2}$$

$$n_i - n_j + n_k = P_{n-4}(s_{ij}); \qquad \tilde{n}_i - \tilde{n}_j + \tilde{n}_k = \tilde{P}_{n-4}(s_{ij})$$

- The polynomials satisfy the constraints [Bjerrum-Bohr, Damgaard, Søndergaard, Vanhove]

$$\sum_\gamma P_{n-4}^\gamma \otimes \tilde{P}_{n-4}^\gamma = 0$$

- Plays some important role in the construction of the integral of the 5-loop amplitude in $\mathcal{N} = 8$ supergravity [Bern et al.]

Part II

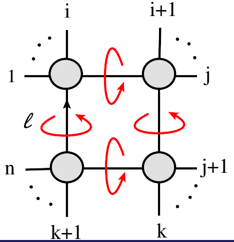
Loop amplitudes

Kinematic relations between loop integral coefficients

At a given loop order the amplitude can be expended on a *finite* basis of (master) integral functions $\mathcal{B}(L)$

$$A^{\rm L-loop} = \sum_{i\in\mathcal{B}(L)} {\rm coeff}_i\, {\rm Integral}_i + {\rm Rational}$$

It is natural to expect that these coefficients satisfy some kinematic relations. One way of getting relations it to consider the maximal cut, cutting the amplitude into products of tree amplitude and using the kinematic relations on the tree



$$= A^{\rm tree}(\ell_1,K_1,-\ell_2)A^{\rm tree}(\ell_2,K_2,-\ell_3)A^{\rm tree}(\ell_3,K_4,-\ell_4)A^{\rm tree}(\ell_4,K_4,-\ell_1)$$

One-loop kinematic monodromy relations I

At one-loop order we can derive monodromy relations generalizing the tree-level construction [Tourkine, Vanhove; Hohenneger, Stieberger]
We start from the representation of the string amplitude with the loop momentum explicit [D'Hoker, Phong]

$$\mathcal{A}(\alpha|\beta) = \int_0^\infty dt \int_{\Delta_{\alpha|\beta}} d^{n-1}\nu \int d^D\ell \, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i}$$
$$\prod_{1\leqslant r<s\leqslant n} f(e^{-2\pi t}, \nu_r - \nu_s) \times e^{-\alpha' k_r\cdot k_s G(\nu_r,\nu_s)}$$

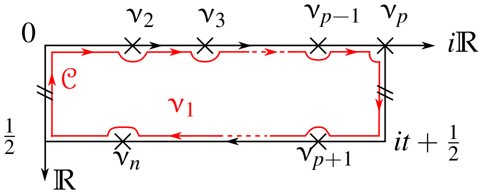
The loop momentum and the Green function without the zero mode

$$\ell^\mu = \int_0^{\frac{1}{2}} d\nu \, \frac{\partial X^\mu(\nu)}{\partial\nu}; \qquad G(\nu_r,\nu_s) = -\log \frac{\vartheta_1(\nu_r - \nu_s|it)}{\vartheta_1'(0)}$$

One-loop kinematic monodromy relations II

We circulate the vertex operator for state 1 inside the planar one-loop integrand (annulus)

$$\oint_{\mathcal{C}} d\nu_1 \int_0^\infty d^D\ell \, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i} e^{-i\pi\alpha'\ell\cdot k_1\,\nu_1} \times \prod_{r=2}^n f(e^{-2\pi t}, \nu_1-\nu_r) e^{-\alpha' k_1\cdot k_r\, G(\nu_1,\nu_r)} = 0$$



The integral vanishes as long as there is no vertex operators in the bulk

One-loop kinematic monodromy relations III

As at tree-level when ν_1 jumps over another vertex operator in the boundary we can phase factor

$$e^{i\alpha' k_1\cdot k_2\, G(\nu_1,\nu_2)} = e^{i\alpha' k_1\cdot k_2 G(\nu_2,\nu_1)}\, e^{\pm i\pi\alpha' k_1\cdot k_2}\,, \tag{1}$$

$$\mathcal{A}(12\cdots m|m+1\cdots n) \to e^{i\pi\alpha' k_1\cdot k_2}\,\mathcal{A}(21\cdots m|m+1\cdots n) \tag{2}$$

When changing boundary $\Re e(\nu_1)\to \Re e(\nu_1)+\frac{1}{2}$ this induces a *loop dependent* phase in the integrand

$$\mathcal{A}(12\cdots n) \to \mathcal{A}(2\cdots n|1)[e^{-i\pi\alpha'\ell\cdot k_1}] :=$$
$$\int_0^\infty dt \int_{\Delta_{2\cdots n|1}} d^{n-1}\nu \prod_{1\leqslant r<s\leqslant n} f(e^{-2\pi t},\nu_r-\nu_s)\, e^{-\alpha' k_r\cdot k_s\, G(\nu_r,\nu_s)}$$
$$\times \int_0^\infty d^D\ell\, e^{-i\pi\alpha'\ell\cdot k_1}\, e^{-\pi\alpha' t\ell^2 - 2i\pi\alpha'\ell\cdot\sum_{k=1}^n k_i\nu_i}$$

One-loop kinematic monodromy relations IV

Circulating ν_1 along the contour of the annulus

$$\mathcal{A}(1,2,\ldots,p|p+1,\ldots,n)+$$
$$\sum_{i=2}^{p-1} e^{i\alpha'\pi k_1\cdot k_{2\ldots i}}\mathcal{A}(2,\ldots,i,1,i+1,\ldots,p|p+1,\ldots,n) =$$
$$-\sum_{i=p}^{n} e^{i\alpha'\pi k_1\cdot k_{p+1\ldots i}}\,\mathcal{A}(2,\ldots,p|p+1,\ldots,i,1,i+1,\ldots,n)[e^{-i\pi\alpha'\ell\cdot k_1}]$$

One-loop kinematic monodromy relations V

In the field theory limit $\alpha' \to 0$ leads to the following planar one-loop amplitude relations

$$A(1\cdots n)[\ell\cdot k_1] + A(21\cdots n)[(\ell+k_2)\cdot k_1] + \cdots + A(23\cdots(n-1)1n)[(\ell+k_{23\ldots n-1})\cdot k_1] = 0$$

- Same expression was derived by [Boels, Isermann] using QFT methods
- These relations are satisfied order by order in the α' expansion. Some analysis of the α' expansion was done by [Hohenneger, Stieberger]

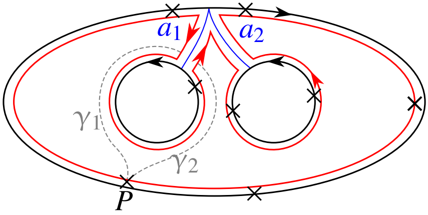
Higher-loop kinematic monodromy relations I

We use again the representation of the amplitude using the loop momenta [D'Hoker, Phong]

$$\int \prod_{i=1}^{g} d\ell_i \, e^{\alpha' i\pi \sum_{I,J} \ell_I \ell_J \Omega_{IJ} - 2i\pi\alpha' \sum_{I,j} \ell_I \cdot k_j \int_P^{z_j} \omega_I}$$

where the loop momenta are defined as the integrals over the a-cycles

$$\ell_I = \int_{a_I} \partial X$$



Higher-loop kinematic monodromy relations II

At two-loop order we get relation between planar and non-planar amplitudes

$$\sum_{r=1}^{|\alpha|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\alpha_s}}\Big)\mathcal{A}^{(2)}(\dots,\alpha_{s-1},1,\alpha_s,\dots|\beta|\gamma)+$$
$$\sum_{r=1}^{|\beta|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\beta_s}}\Big)\mathcal{A}^{(2)}(\alpha|\dots,\beta_{s-1},1,\beta_s,\dots|\gamma)[e^{-i\alpha'\pi\ell_1\cdot k_1}]+$$
$$\sum_{r=1}^{|\gamma|}\Big(\prod_{s=1}^{r} e^{i\alpha'\pi k_1\cdot k_{\gamma_s}}\Big)\mathcal{A}^{(2)}(\alpha|\beta|\dots,\gamma_{s-1},1,\gamma_s,\dots)[e^{-i\alpha'\pi\ell_2\cdot k_1}]$$
$$=0$$

The monodromy relations will always mix planar and non-planar amplitudes

Higher-loop kinematic monodromy relations III

At four points we get this reads

$$\mathcal{A}^{(2)}(1234) + e^{i\pi\alpha' k_1\cdot k_2}\mathcal{A}^{(2)}(2134) + e^{i\pi\alpha' k_1\cdot k_{23}}\mathcal{A}^{(2)}(2314) + \mathcal{A}^{(2)}(234|1|.)[e^{-i\pi\alpha'\ell_1\cdot k_1}] + \mathcal{A}^{(2)}(234|.|1)[e^{-i\pi\alpha'\ell_2\cdot k_1}] = 0$$

where $\mathcal{A}^{(2)}(1234)$ etc. are planar two-loop amplitude integrands, and $\mathcal{A}^{(2)}(234|1|.), \mathcal{A}^{(2)}(234|.|1)$ are the two non-planar amplitude integrands with the external state 1 on the b_I-cycle with $I = 1, 2$

The field theory limit of that relation, at leading order in α', leads to

$$A^{(2)}(1234) + \mathcal{A}^{(2)}(2134) + A^{(2)}(2314) + A^{(2)}(234|1|.) + A^{(2)}(234|.|1) = 0$$

This matches the expression derived in QFT by [Feng, Jia, Huang]

Outlook

Using string theory we have extended the kinematic relations between amplitudes in string theory and quantum field theory

- These relations are independent of any particular numerator (BCJ) representation of the integrand of the multiloop amplitudes
- They provide model independent kinematic relations (true for independent of the nature of external legs, do not rely on supersymmetry, ...)
- At loop orders contact terms are needed to match the structure of the multiloop amplitudes gauge and gravity
- Interesting new relations on integral coefficients [work in progress with Tourkine and Ochirov]