Three-loop massive tadpoles and the polylogarithms up to weight six

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In collaboration with A.F. Pikelner and B.A. Kniehl

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A9 project: Loops and Legs

- A9 project makes use of the natural synergy between directly experimentally relevant calculations and theoretically motivated considerations to develop the new technology toward maturity.
- Members:
  - Bernd Kniehl
  - Rutger Boels
  - Gustavo Alvarez Cosque
  - Mikhail Kalmykov
  - Hui Luo
  - Oleg Veretin
- Former members: Isabella Bierenbaum, Paolo Bolzoni, Tobias Hansen, Christoph Horst, Reinke Isermann, Arthur Lipstein, Oleg Tarasov, Gang Yang

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- develope and test the method based both on differential equations and asymptotic expansions for the evaluation of the Feynman integrals

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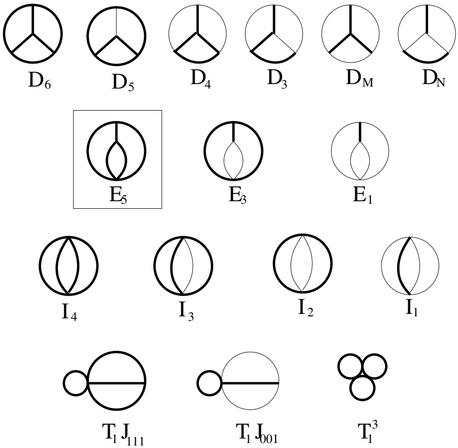
- as a first application we apply it to the 3-loop massive vacuum bubble integrals; these are relevant in the following evaluations, e.g.
- evaluation of the EW vacuum potential beyond NNLO:
  (currently full 2-loop EW corrections are know + 3-loop QCD)
  $\Longrightarrow$ project B4
- matching of the running parameters and the observables:
  needed for the analysis of the vacuum stability
  $\Longrightarrow$ project B4, (talk of A. Pikelner)
- bubble integrals serve as a boundary condidtions for more complicated objects
- deeper ε-expansion of the 3-loop bubbles is a necessary step toward 4-loop calculations

single-scale master integrals



- reduction $\longrightarrow$ MATAD Steinhauser '01
- weight 4 hyperlogarithms result Broadhurst '99

the plan

- introduce two different masses and the ration $z = m_1^2/m_2^2$
- using IBP identities and write a system of differential equations in z
- find all boundary conditions, using asymptotic expansion as $z \to 0$
- solve the system of differential equations as a generalized power series
- if possible solve as hypergeometric series, if not
- set $z = 1$ and sum up numerically with multiple precision
- apply PSLQ algorithm to get analytical results for master integrals (this requires to know the relevant irrational constants that form basis over Q)

differential equations

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- substitute the Ansatz

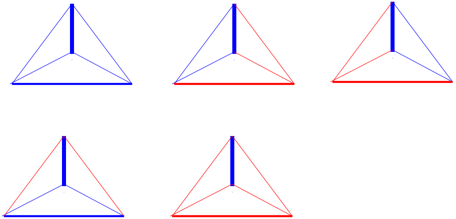
$$f_j = \sum_{n=0}^{\infty} \sum_{k=1}^{K} c_{j,n,k} z^{\mu_k + n}$$

The exponent shifts μ_k are determined as usual in the Frobenius method from the indicial polynomials. Actually, it is easy to establish that μ_j can take values $0, -\varepsilon, -2\varepsilon, -3\varepsilon$. Therefore we have for each j four different solutions $f_j^{(1)}, f_j^{(2)}, f_j^{(3)}, f_j^{(4)}$, corresponding to different values of μ and the solution we are looking for is the linear combination

$$f_j = \sum_{k=1}^{4} C_{j,k} f_j^{(k)}$$

boundary conditions

- we cannot set directly $z \to 0$: infrared singularity!
- use large mass expansion in $1/M^2$ instead
- example: $\mathbf{D_N}$ integral



- Different subgraph have different scaling with $M^{a\varepsilon}$, $a = 0, -1, -2, -3$

summation of the series

- if there is a threshold at $z = 1$ the convergence of the series $\sum c_n z^n$ is governed by the asymptotics

$$c_n \sim \frac{1}{n^\alpha} \qquad \text{as} \quad n \to \infty$$

- accelerating methods, based on series transformations
- if the asymptotic is known, the new integral methods become available

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with some α_j and C_j depending on d

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$$c_n \sim \frac{C_1}{n^{\alpha_1}}\left(1+\frac{y_{1,1}}{n}+\ldots\right) + \cdots + \frac{C_k}{n^{\alpha_k}}\left(1+\frac{y_{k,1}}{n}+\ldots\right)$$

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- C_j require new information

asymptotic solution of recurrence

- Consider homogenious recurrence of the the order K

$$p_0(n)c_n + p_1(n)c_{n+1} + \cdots + p_K(n)c_{n+K} = 0\,, \tag{1}$$

here $p_k(n)$ are the polynoms in $1/n$ of degree h:

$$\frac{p_k(n)}{n^h} = b_0^{(k)} + \frac{b_1^{(k)}}{n} + \frac{b_2^{(k)}}{n^2} + \cdots + \frac{b_h^{(k)}}{n^h}\,, \tag{2}$$

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- In order to find λ_j and α_j substitute the Ansatz (3) into recurrence (1) with the coefficients (2).

asymptotic solution of recurrence(2)

Introduce sums:

$$B_m^{(0)}(\lambda) = \sum_{k=0}^{K} \lambda^k b_m^{(k)}, \qquad 0 \le m \le h\,,$$

$$B_m^{(1)}(\lambda) = \lambda \frac{dB_m^{(0)}}{d\lambda} = \sum_{k=0}^{K} k\lambda^k b_m^{(k)}\,,$$

$$B_m^{(2)}(\lambda) = \lambda \frac{dB_m^{(1)}}{d\lambda} = \sum_{k=0}^{K} k^2\lambda^k b_m^{(k)}\,, \qquad \text{etc.}$$

Then λ_j, $j = 1, \dots, K$ are the roots of

$$B_0^{(0)} = 0$$

asymptotic solution of recurrence(3)

Next we find α_j, $j = 1, \ldots, K$
(notation: $[\alpha]_m = a(a-1)\ldots(a-m+1) = \Gamma(\alpha+1)/\Gamma(\alpha-m)$)

simple root, then coresponding α is the root of

$$[\alpha]_1 B_0^{(1)} + B_1^{(0)} = 0$$

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triple root, then coresponding $\alpha_{1,2,3}$ are the roots of

$$\frac{[\alpha]_3}{3!} B_0^{(3)} + \frac{[\alpha]_2}{2!} B_1^{(2)} + \frac{[\alpha]_1}{1!} B_2^{(1)} + B_3^{(0)} = 0$$

...etc

polylogarithms

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Goncharov's polylogarithms

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{i_1>i_2>\cdots>i_k>0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_1^{i_k}}{i_1^{m_k}}$$

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generalized multiple polylogarithms

$$G_{a_1 a_2 \ldots a_w}(z) = \int_0^z \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_1}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}, \qquad a_w \neq 0,$$

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relation between the two above

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k G_{\underbrace{0,\ldots,0}_{m_1-1}\frac{1}{x_1},\ldots,\underbrace{0,\ldots,0}_{m_k-1}\frac{1}{x_1 x_2 \ldots x_k}}(1)$$

polylogarithms (2)

$$G_{a_1 a_2 \dots a_w} = \int_0^1 \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_1}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}\,, \qquad a_w \neq 0\,,$$

- w is called the weight of the polylogarithm
- form algebra w.r.t. shuffle product

$$G_{a_1 \dots a_{w_1}} G_{b_1 \dots b_{w_2}} = \sum_{c \in a \uplus b} G_{c_1 \dots c_{w_1 + w_2}}$$

- shtuffle product (written in dual representation)

algebra of the sixht root of unity

- let $\omega = e^{i\frac{\pi}{3}}$ is the sixth root of unity
- elements $G_{a_1 \ldots a_w}$ where a_j are taken from 7-letter alphabet $\{0, \omega^0, \omega^1, \omega^2, \ldots, \omega^5\}$ form algebra of the sixth root of unity

$$\mathcal{A} = \oplus_{j=1}^{\infty} \mathcal{A}_j$$

- relevance for the massive Feynman integrals Broadhurst '99

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- some subalgebras
  - alphabet $\{0, 1\} \longrightarrow$ Euler–Zagier sums (EZ)
  - alphabet $\{0, 1, -1\} \longrightarrow$ multiple zeta values (MZV)
  - alphabet $\{0, -1, \omega^5\} \longrightarrow$ Deligne numbers

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  - alphabet $\{0, -1, \omega^5\} \longrightarrow$ Deligne numbers
- we want to find the set of constants that
  - much smaller than $\mathcal{A}$
  - includes all 3-loop vacuum integral to all order in ε
  - is an algebra

construction of $\mathcal{A}_{H(\omega)}$

Harmonic polylogarithms of the sixth root of unity

$$H_{n_1 n_2 \dots n_w}(\omega) = \int\limits_0^{\omega} \frac{dt_1}{t_1 - n_1} \int\limits_0^{t_1} \frac{dt_1}{t_2 - n_2} \cdots \int\limits_0^{t_{w-1}} \frac{dt_w}{t_w - n_w}\,, \qquad n_j = 0, +1, -1\,,$$

- has the desired properties
- e.g. $|\mathcal{A}_6| = 117649$ while $|\mathcal{A}_{H(\omega),6}| = 729$
- $\Re\mathcal{A}_{H(\omega)}$ is isomorph to $\mathcal{A}_{H(\omega^2)}$
- split each element into real and imaginary parts $\longrightarrow \Re\mathcal{A}_{H(\omega)}, \Im\mathcal{A}_{H(\omega)}$

construction of $\mathcal{A}_{H(\omega)}$

Dimensions of the $\mathcal{A}_J$

w	$\Re\mathcal{A}$	$\Im\mathcal{A}$	$\Re\mathcal{A}_{H(\omega)}$	$\Im\mathcal{A}_{H(\omega)}$
1	2	1	1	1
2	5	3	3	3
3	12	9	8	8
4	30	25	21	21
5	76	68	55	55
6	195	182	144	144

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Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

uniform weight representation for the master integrals

For all **D** integrals (mercedez)

$$\mathbf{D}_x = \frac{1}{(1-\varepsilon)(1-2\varepsilon)}\left(\frac{2\zeta_3}{\varepsilon} + \bar{D}_x^{(0)} + \varepsilon \bar{D}_x^{(1)} + \varepsilon^2 \bar{D}_x^{(2)} + \dots\right),$$

where $\bar{D}_x^{(J)}$ are uniform of weights $J+4$

$$\begin{aligned}
\bar{D}_6^{(0)} =& -\frac{72}{11}\mathcal{R}_{1,-1,1,0} + \frac{180}{11}\mathcal{R}_{1,-1,1,1} + \frac{148}{11}\mathcal{R}_{1,0,1,0} - \frac{144}{11}\mathcal{R}_{1,1,-1,0} \\
&+ \frac{360}{11}\mathcal{R}_{1,1,-1,1} + \frac{540}{11}\mathcal{R}_{1,1,1,-1} - \frac{33587}{55}\mathcal{R}_{1,1,1,1} \\
\bar{D}_6^{(1)} =& 156\mathcal{R}_{1,-1,1,1,0} - 16\mathcal{R}_{1,0,-1,1,0} - 16\mathcal{R}_{1,0,1,-1,0} - 468\mathcal{R}_{1,0,1,1,1} \\
&+ \frac{7712}{87}\mathcal{R}_{1,1,-1,0,0} - \frac{7084}{87}\mathcal{R}_{1,1,-1,0,1} + \frac{15884}{87}\mathcal{R}_{1,1,-1,1,0} \\
&+ \dots
\end{aligned}$$

conclusions

- we develope the method of eveluation of Feynman diagrams, based on the differential equations and IBP
- the boundary conditions are take from the asymptotic expansion in the large mass regime
- we construct the set of constants $\mathcal{A}_{H(\omega)}$ that form an algebra w.r.t. shuffle/shtuffle relation
- we evaluate all the three-loop massive tadpole diagrams analytically in terms of the above constants
- we give the representation in terms of uniform basis for all order of ε-expansion